ASYMPTOTIC SPECTRAL FLOW FOR DIRAC OPERATORS OF DISJOINT DEHN TWISTS

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ABSTRACT. Let Y be a compact, oriented 3-manifold with a contact form a. For any Dirac operator $\mathcal{D}$, we study the asymptotic behavior of the spectral flow between $\mathcal{D}$ and $\mathcal{D}+\mathrm{cl}(-\frac{ir}{2}a)$ as $r\to\infty$. If a is the Thurston–Winkelnkemper contact form whose monodromy is the product of Dehn twists along disjoint circles, we prove that the next order term of the spectral flow function is $\mathcal{O}(r)$.

1. Introduction

1.1. **Asymptotic spectral flow.** For a pair of purely-imaginary valued 1-forms, A_0 and A_1, choose a path of 1-forms, $A(s)$, connecting A_0 to A_1. For a Dirac operator $\mathcal{D}$, consider the family of Dirac operators, $\{\mathcal{D}_{A(s)} = \mathcal{D}+\mathrm{cl}(\frac{A(s)}{2})\}_{s\in[0,1]}$, which is $\mathcal{D}$ perturbed by the Clifford action of $A(s)$. The eigenvalues of each $\mathcal{D}_{A(s)}$ are unbounded from above and below, and vary continuously along the path. The *spectral flow* is the algebraic count of the zero crossings: a zero crossing contributes to the count with $+1$ if the eigenvalue crosses zero from a negative to a positive value as t increases, and count with -1 if the eigenvalue crosses zero from a positive to a negative value as s increases. If the path is suitably generic, only these two cases would arise. Moreover, the count turns out to be path independent ([APS]), and so depends only on the ordered pair $(\mathcal{D}_{A_0},\mathcal{D}_{A_1})$. This count is the spectral flow function.

If we have a real valued 1-form a, we can consider the spectral flow with $A_0=0$ and $A_1=-\frac{ir}{2}a$. The spectral flow can be thought as a function of r, and we denote it by $\mathrm{sf}_a(\mathcal{D},r)$. In [T2, section 5] and [T3], Taubes studied the asymptotic behavior of the spectral flow function as $r\to\infty$. He proved:

Theorem 1.1 ([T2])**.** *There exists constants $\delta\in(0,1/2)$ and c with the following significance: Suppose that Y is a compact, oriented 3-manifold and $\mathcal{D}$ is a Dirac operator on Y. For any real-valued 1-form a with $||a||_{\mathcal{C}^2}\le 1$, the spectral flow function satisfies*

$$\big|\mathrm{sf}_a(\mathcal{D},r)-\frac{r^2}{32\pi^2}\int_Y a\wedge \mathrm{d}a\big|\le c\,r^{\frac{3}{2}+\delta} \tag{1.1}$$

for all $r\ge c$.

This theorem gives the leading order term of the spectral flow function and gives a bound on the next order term.

It follows from the results in the seminal paper by Atiyah, Patodi and Singer [APS] that the spectral flow gives the index of an associated Dirac operator. To give the basic idea, we briefly explain the finite dimensional case: Suppose that there are two non-degenerate Hermitian matrices, H_- and H_+. Connect them by a path of Hermitian matrices, $H(s)$ with $H(s) = H_-$ for $s \leq -1$ and $H(s) = H_+$ for $s \geq 1$. The zero crossings of the eigenvalues of $H(s)$ is the spectral flow for H_- and H_+. It is easy to see that the spectral flow only depends on H_- and H_+, but not on the path $H(s)$. With $H(s)$, define the operator $\mathfrak{D} : \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^m) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^m)$ by

$$\mathfrak{D} = \partial_s + H(s)$$

where s is the coordinate on $\mathbb{R}$, and m is the size of the Hermitian matrices. This operator $\mathfrak{D}$ is a Fredholm operator, and its index is given by the spectral flow between H_- and H_+. These notions also fit with the Morse theory. If we take $H(s)$ to be the Hessian of the Morse function along a gradient flow line between two critical points, the index of $\mathfrak{D}$ gives the expected dimension of the space of gradient flow lines. Meanwhile, the spectral flow for H_- and H_+ is exactly the difference between the Morse index of the two critical points.

1.2. **Next order term on contact 3-manifold.** Theorem 1.1 is established for any 1-form. According to [APS], the next order term of $\mathrm{sf}_a(\mathcal{D}, r)$ is essentially the η-invariant of the Dirac operator $\mathcal{D}_{-ira}$, see subsection 2.4 for the detail. Unlike the spectral flow function, it only depends on the gauge equivalent class of $\mathcal{D}_{-ira}$, but not the path of connections. (It would be interesting to estimate the η-invariant of $\mathcal{D}_{-ira}$ directly.) This paper looks at the size of the next order term when a is a *contact form*.

Question. For a contact form a with an adapted Riemannian metric, is the next order term of the spectral flow function $\mathrm{sf}_a(\mathcal{D}, r)$ of order $\mathcal{O}(r)$ as $r \to \infty$?

The construction of Dirac operators requires a Riemannian metric. We always choose an adapted metric to make it easier to compare the spectral flow function. On a contact 3-manifold, an *adapted Riemannian metric* is a metric such that $|a| = 1$ and $\mathrm{d}a = 2 * a$. The existence of such metric is proved by [CH]. In this paper, we give an affirmative answer to the question in following situation:

Main Theorem (Theorem 2.1)**.** *Suppose that the monodromy of an open book decomposition is the product of Dehn twists along disjoint circles, and a is the associated Thurston–Winkelnkemper contact form. Then, with certain adapted metric, the next order term of the spectral flow function of the canonical Dirac operator $\mathcal{D}$ is of order $\mathcal{O}(r)$. Namely, there exists a constant c such that*

$$\left|\mathrm{sf}_a(\mathcal{D}, r) - \frac{r^2}{32\pi^2} \int_Y a \wedge \mathrm{d}a\right| \leq cr.$$

for all $r \geq 1$.

What follows explains why such bound on the next order term is expected: As $r \to \infty$, the zero modes of the Dirac operators $\mathcal{D}_{-ira}$ have the following properties: Their derivative along the direction of the Reeb vector field is close to the multiplication by $r/2$. On the contact hyperplane, they almost satisfy the Cauchy–Riemann equation. In this regards, the equation is very similar to the almost holomorphic equation in [D] and [IMP]. This being the case, the general case looks locally like the circle bundle case. Suppose that Y is a $U(1)$-bundle over a Riemann surface with nontrivial Euler class, and a is a connection 1-form with constant curvature. Its spectral flow function can be computed by the Riemann–Roch formula, and the next order term is of order $\mathcal{O}(r)$.

This paper is organized as follows:

Section 2 provides the background for this paper. Subsection 2.1 sets up the conventions of Dirac operators on contact 3-manifold and spectral flow functions. Subsection 2.2 reviews the open book decompositions and the constructions of Thurston–Winkelnkemper contact forms.

Section 3 contains some basic estimates on the zero modes of $\mathcal{D}_{-ira}$. Proposition 3.1 is the cornerstone of all the other estimates in this paper.

Section 4 writes down the Dirac equation on different regions of the open book decomposition. For the tubular neighborhood of the binding and the Dehn-twist region, subsection 4.3 describes a procedure to compactify them as $S^2 \times S^1$. On where the monodromy is identity, proposition 4.4 constructs almost zero modes of the Dirac operator.

Section 5 studies the Dirac equation in the model case, $S^2 \times S^1$. There are two main points in this section: The first one is subsection 5.2, which builds an expansion of the zero modes of the Dirac operator in this model case. The second one is lemma 5.14, which cuts the interval $[0, r]$ into subintervals such that there are sparse zero crossings near the endpoints of the subintervals.

Section 6 combines above results to prove theorem 6.3, the lower bound of the spectral flow function.

Section 7 is a further study of the Dirac equation on the region where the monodromy is the identity map. There is a S^1-action in this region. Lemma 7.1 and proposition 7.2 shows that there is only one frequency such that a zero mode can have significantly large component. Lemma 7.3 characterizes the behavior of that component near the boundary of this region.

The last section proves the upper bound of the spectral flow function. Unlike the eigensections of a fixed Dirac operator, zero modes of a family of Dirac operators are not necessarily orthogonal to each other. The whole section is devoted to overcome this difficulty.

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2. Preliminary

In this section, we set up the background on the spectral flow function and the open book decomposition.

2.1. **Spectral flow of contact form.**

We now review the Dirac operator and its spectral flow. We focus on the canonical Dirac operator perturbed by the contact form, and will not do the construction of general $\mathrm{Spin}^{\mathbb{C}}$ structures. Below we mostly follow [T2].

2.1.1. *Canonical* $\mathrm{Spin}^{\mathbb{C}}$ *structure.* Let Y be a closed oriented connected 3-manifold with a contact form a. Fix an adapted Riemannian metric on Y. The Reeb vector field, which we denote by e_3, has unit length measured by any adapted metric.

Consider the contact hyperplane field $\ker(a) \subset TY$. With the orientation given by $\mathrm{d}a$, it is a Hermitian line bundle on Y. We denote it by K^{-1}, and its inverse bundle K is called the *canonical line bundle.* The bundle $\underline{\mathbb{C}} \oplus K^{-1}$ has a Clifford action

$$\mathrm{cl} : TY \longrightarrow \mathrm{End}(\underline{\mathbb{C}} \oplus K^{-1})$$

which is defined as follows: Fix a unit normed, trivializing section of $\underline{\mathbb{C}}$, and denote it by $\mathbf{1}_{\underline{\mathbb{C}}}$. For any oriented orthonormal frame on the contact hyperplane $\{e_1, e_2\}$, it gives a unit normed, trivializing section of K^{-1}, $\frac{1}{\sqrt{2}}(e_1 - ie_2)$. With this trivialization, the Clifford action is given by the Pauli matrices

$$\mathrm{cl}(e_1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \mathrm{cl}(e_2) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \qquad \mathrm{cl}(e_3) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

The bundle $\underline{\mathbb{C}} \oplus K^{-1}$ together with this Clifford action is called the *canonical* $\mathrm{Spin}^{\mathbb{C}}$ *structure.*

2.1.2. *Dirac operator.* A $\mathrm{Spin}^{\mathbb{C}}$ *connection* on the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure is a Hermitian connection ∇_A on $\underline{\mathbb{C}} \oplus K^{-1}$ which is compatible with the Clifford action in the following sense: If v is a vector field and ψ is a section of $\underline{\mathbb{C}} \oplus K^{-1}$, then

$$\nabla_A(\mathrm{cl}(v)\psi) = \mathrm{cl}(\nabla v)\psi + \mathrm{cl}(v)\nabla_A\psi.$$

Here ∇v is the covariant derivative of v with respect to the Levi-Civita connection on TY.

Given a $\mathrm{Spin}^{\mathbb{C}}$ connection, define the *Dirac operator* to be the composition

$$\mathcal{C}^\infty(\underline{\mathbb{C}} \oplus K^{-1}) \xrightarrow{\nabla_A} \mathcal{C}^\infty(T^*Y \otimes (\underline{\mathbb{C}} \oplus K^{-1})) \xrightarrow{\mathrm{cl}} \mathcal{C}^\infty(\underline{\mathbb{C}} \oplus K^{-1})$$

The Clifford action is extended to the cotangent bundle using the Hodge star operator.

It is an easy exercise to show that there exists a unique $\mathrm{Spin}^{\mathbb{C}}$ connection such that the section $(\mathbf{1}_{\underline{\mathbb{C}}}, 0)$ is annihilated by the associated Dirac operator. This connection is called the *canonical connection*. We denote the associated Dirac operator by D_0. In this paper, we focus on the following family of Dirac operators: For each $r \geq 0$,

$$D_r = D_0 + \mathrm{cl}(\frac{-ir}{2}a). \tag{2.1}$$

Note that $\mathrm{cl}(a)$ acts as i on the $\underline{\mathbb{C}}$-summand and as $-i$ on the K^{-1}-summand.

2.1.3. *Spectral flow function.* We now define the spectral flow function for the family of Dirac operators $\{D_s\}_{s\in[0,r]}$. The construction is borrowed from [T2, subsection 5.1].

The spectral flow for the family $\{D_s\}_{s\in[0,r]}$ is defined with the help of a certain stratified, real-analytic set in $\mathbb{R} \times [0, r]$. This set is denoted by $\mathcal{E}$, and its stratification is given by

$$\mathcal{E} = \mathcal{E}_1 \supset \mathcal{E}_2 \supset \cdots ,$$

where $\mathcal{E}_l$ of the set of pairs (λ, s) such that λ is an eigenvalue of D_s with multiplicity l or greater. Each $\mathcal{E}_l$ is a closed set. Moreover, as can be proved using the results in [K, chapter 7], each $\mathcal{E}_{l^*} = \mathcal{E}_l - \mathcal{E}_{l+1}$ is an open and real analytic submanifold of $\mathbb{R}\times[0, r]$. The collection $\{\mathcal{E}_{l^*}\}$ are called the *smooth strata* of $\mathcal{E}$. When the 1-dimensional smooth strata are oriented by the pull-back from $\mathbb{R} \times [0, r]$ of the 1-form $\mathrm{d}s$, then the zero dimensional strata can be consistently oriented so that the formal, weighted sum $\mathcal{E}_* = \sum_{l\in\mathbb{N}} \mathcal{E}_{l^*}$ defines a locally closed cycle in $\mathbb{R} \times [0, r]$. It also follows from the results in [K, chapter 7]. It means the following: Let h be a smooth function on $\mathbb{R} \times (0, r)$ with compact support, then

$$\sum_{l\in\mathbb{N}} \int_{\mathcal{E}_{l^*}} \mathrm{d}h = 0$$

Sard's theorem finds a dense, open set $\mathbb{U} \subset \mathbb{R}$ with the property that the respective maps from a point, $\star$, to $\mathbb{R}\times[0, r]$ that sends $\star$ to $(\lambda, 0)$ and to (λ, r) are transverse to the smooth strata of $\mathcal{E}$ for all $\lambda \in \mathbb{U}$. With this understood, the spectral flow for $\{D_s\}_{s\in[0,r]}$ is defined as the follows: Fix $\lambda_0 \in \mathbb{U}$ and $\lambda_0 > 0$. By Sard's theorem, there exist smooth, oriented paths $\sigma \subset \mathbb{R}\times[0, r]$ that starts at $(\lambda_0, 0)$, and ends at (λ_0, r), and are transverse to the smooth strata of $\mathcal{E}$. Such a path has the following well-defined intersection number with $\mathcal{E}$

$$\mathrm{sf}_a(r, \lambda_0) = \sum_{l\in\mathbb{N}} \sum_{p\in\sigma\cap\mathcal{E}_{l^*}} \iota(p)l,$$

where $\iota(p) \in \{-1, 1\}$ is the sign of intersection. In the case where σ is the graph of a smooth function from $[0, r]$ to $\mathbb{R}$, the sign $\iota(p)$ is obtained

as follows: The pull-back to a smooth, 1-dimensional stratum of $\mathcal{E}$ of the 1-form $\mathrm{d}\lambda$ from $\mathbb{R}\times[0,r]$ at a point (λ,u) can be written as $\lambda'\mathrm{d}u$ with

$$\lambda' = \int_Y \langle\psi, \mathrm{cl}(\frac{-i}{2}a)\psi\rangle,$$

where ψ is a unit length eigensection of D_u with eigenvalue λ. The sign of λ' at an intersection point with the image of a graph σ is the factor $\iota(p)$.

If λ_0 is sufficiently close to 0, $\mathrm{sf}_a(r,\lambda_0)$ is independent of λ_0. The *spectral flow function* for $\{D_u\}_{u\in[0,r]}$ is defined to be

$$\mathrm{sf}_a(r) = \lim_{\lambda_0\to 0^+} \mathrm{sf}_a(r,\lambda_0). \tag{2.2}$$

2.2. **Open book decomposition.** We now briefly review the open book decomposition and contact 3-manifold. For a complete discussion, consult [OS, chapter 9] or [E] and the references therein. The notations introduced in this section will be used throughout the paper.

2.2.1. *Open book decomposition.* An *(abstract) open book decomposition* consists of a Riemann surface $\bar{\Sigma}$ with boundaries $\{C_j\}_{j\in J}$, and a self-diffeomorphism τ which is identity near the boundary. The 3-manifold Y is obtained by the following construction: at first, form the mapping torus

$$\bar{\Sigma}\times_\tau S^1 = \frac{\bar{\Sigma}\times[0,2\pi]}{(p,2\pi)\sim(\tau(p),0)}.$$

Its boundary is the disjoint union of tori $\coprod_j C_j\times S^1$. We then attach solid tori $\coprod_j S^1\times D^2$ to it, where the longitude is identified with the C_j-factor, and the meridian is identified with the S^1-factor. Note that there is a S^1-family of $\bar{\Sigma}$ in Y, and they are referred as the *pages*. The cores of the attaching solid tori are called the *bindings*.

In next subsection, the handle attaching will be described explicitly in terms of local coordinates.

2.2.2. *Contact form.* Given an open book decomposition, Thurston and Winkelnkemper [TW] construct a contact form a on it. The contact form a has the following significance: On the mapping torus $\bar{\Sigma}\times_\tau S^1$, the Reeb vector field is transverse to the pages, and $\mathrm{d}a$ restricted to the pages is an area form. On the attaching solid tori, a is of certain standard form, and the bindings are Reeb orbits.

By the celebrated work of Giroux [G], every contact form is isotopic to a contact form of this sort.

We now describe the contact form. We first do the case when the monodromy is the identity map, then the case when the monodromy is the product of Dehn twists along disjoint circles. We will not discuss the general case. In what follows, ϵ is a fixed constant less than 0.01.

With trivial monodromy. If the monodromy is the identity map, the mapping torus is $\bar{\Sigma} \times S^1$. Near each boundary circle C_j, let ρe^{it} be the coordinate on its collar neighborhood where $\rho \in [1, 1+20\epsilon)$ and $C_j = \{\rho = 1\}$. Choose a 1-form $\mu_{\bar{\Sigma}}$ on $\bar{\Sigma}$ satisfying

- $\mathrm{d}\mu_{\bar{\Sigma}}$ is an area form on $\bar{\Sigma}$;
- $2\mu_{\bar{\Sigma}} = (2-\rho)\mathrm{d}t$ on the collar neighborhood of C_j.

See [OS, p.141] for the existence of such $\mu_{\bar{\Sigma}}$. Let $e^{i\phi}$ be the coordinate on the S^1-component of $\bar{\Sigma} \times S^1$. The contact form on this region, $\bar{\Sigma} \times S^1$, is

$$a = V\mathrm{d}\phi + 2\mu_{\bar{\Sigma}} \tag{2.3}$$

where V is a constant greater than 1. We will adjust V to be a larger constant later.

Attaching handles. Let $(e^{it}, \rho e^{i\phi})$ be the coordinate on the attaching solid tori $S^1 \times D^2$ where $\rho e^{i\phi}$ is the polar coordinate and $\rho \le 1$. The handle attaching is exactly by identifying the coordinates. Choose two smooth function f and g which only depend on ρ and

- When $\rho \in [1-5\epsilon, 1]$, the function f is V, and g is $2-\rho$.
- When $\rho \in [0, 10\epsilon]$, the function f is ρ^2, and g is $2-\rho^2$.
- $f'(\rho) \ge 0$, and $g'(\rho) < 0$ except at $\rho = 0$.

If we think the construction as drawing a curve $(f(\rho), g(\rho))$ on $\mathbb{R}^2$ satisfying those conditions, the existence of f and g is clear. We can take

$$a = f(\rho)\mathrm{d}\phi + g(\rho)\mathrm{d}t \tag{2.4}$$

to be the contact form on the attaching solid tori.

Disjoint Dehn twists. If the monodromy is the product of Dehn twists along disjoint circles $\{\Gamma_l\}$, we only need to modify the contact form near these circles.

Note that $\mathrm{d}\mu_{\bar{\Sigma}}$ is a symplectic form, and Γ_l is a Lagrangian submanifold. By the Weinstein tubular neighborhood theorem, there exists coordinate ρe^{it} on a tubular neighborhood of Γ_l such that $\mathrm{d}\mu_{\bar{\Sigma}} = \mathrm{d}t \wedge \mathrm{d}\rho$. Without loss of generality, we may assume that $\rho \in (-1-15\epsilon, 1+15\epsilon)$ and $\Gamma_l = \{\rho = 0\}$. By adding the differential of a smooth function, we can assume that $\mu_{\bar{\Sigma}}$ is $2(v_l - \rho)\mathrm{d}t$ on the tubular neighborhood of Γ_l. The period constant $2v_l = \int_{\Gamma_l} \mu_{\bar{\Sigma}}$ is given by the original choice of $\mu_{\bar{\Sigma}}$.

To perform the Dehn twist along Γ_l, choose a smooth function $\tau_l(\rho)$ satisfying

$$\tau_l(\rho) = \begin{cases} 0 & \text{when } \rho \le -1+5\epsilon \\ \pm 2\pi N_l & \text{when } \rho \ge 1-5\epsilon \end{cases}$$

where the plus/minus sign corresponds to the right-handed/left-handed (positive/negative) Dehn twist, and N_l is a positive integer corresponding to the

power of the Dehn twist. The mapping torus of the tubular neighborhood of Γ_l is

$$\frac{(-1-15\epsilon, 1+15\epsilon)\times S^1\times[0,2\pi]}{(\rho, e^{i(t+\tau_l(\rho))}, 0)\sim(\rho, e^{it}, 2\pi)}.$$

Let ϕ be the coordinate on the interval $[0,2\pi]$. If we take V to be a large constant such that

$$V\geq 1+2\big|(v_l-\rho)^2\tau_l'(\rho)\big|+2\big|(v_l-\rho)\tau_l(\rho)\big| \tag{2.5}$$

for $|\rho|<1+15\epsilon$, the 1-form

$$a=V\mathrm{d}\phi+2(v_l-\rho)\mathrm{d}t+2\phi(v_l-\rho)\tau_l'(\rho)\mathrm{d}\rho \tag{2.6}$$

is a contact form which coincides with (2.3) when $|\rho|>1$. More precisely, (2.6) is a contact form on $(-1-15\epsilon,1+15\epsilon)\times S^1\times[0,2\pi]$, and is invariant under the identification map. The condition (2.5) guarantees something more than being a contact form. Subsection 5.5 is a remark about this condition.

The 3-manifold will be denoted by Y. Let Σ be the surface obtained by cutting out the tubular neighborhood of Γ_l, $\{|\rho|<1\}$, from $\bar{\Sigma}$. The monodromy is identity on Σ. The restriction of $\mu_{\bar{\Sigma}}$ is denoted by μ_Σ.

2.3. **Spectral flow estimate.** With the above construction, here is the precise statement of our main result:

Theorem 2.1. *If the monodromy of the open book decomposition is the product of Dehn twists along some disjoint circles, consider the contact form a given by (2.3), (2.4) and (2.6). Then, there exist an adapted Riemannian metric and a constant c such that*

$$\Big|\mathrm{sf}_a(r)-\frac{r^2}{32\pi^2}\int_Y a\wedge\mathrm{d}a\Big|\leq cr$$

for all $r\geq 1$.

The proof is sort of the gluing construction. The arguments for each Dehn-twist region will be the same. We will treat it as there is only one Dehn twist, and the subscription l will be dropped. For the same reason, we will treat it as there is only one binding in the open book decomposition. For more than one Dehn twist or binding, one only need to put union/summation in the argument. It only boosts the complexity of the notations, and we will not do that.

2.4. **Estimate on the eta invariant.** As pointed out in [T3, p.573], theorem 2.1 has a corollary concerning the eta invariant of the Dirac operator D_r. As the background for the corollary, consider the 4-manifold $X=[0,1]\times Y$, and let s be the coordinate on $[0,1]$. Give X the product Riemannian metric. The projection from X to Y pulls back $\underline{\mathbb{C}}\oplus K^{-1}$ to give a $\mathbb{C}^2$-bundle over X, which is also denoted by $\underline{\mathbb{C}}\oplus K^{-1}$.

Choose a real-valued smooth function $\nu(s)$ on $\mathbb{R}$ with $\nu(s) = 0$ when $s \leq \epsilon$ and $\nu(s) = 1$ when $s \geq 1 - \epsilon$. With $\nu(s)$ chosen, define the operator

$$\begin{array}{rccc} \mathfrak{D}_r : & \mathcal{C}^\infty(X; \underline{\mathbb{C}} \oplus K^{-1}) & \to & \mathcal{C}^\infty(X; \underline{\mathbb{C}} \oplus K^{-1}) \\ & \psi & \mapsto & \partial_s \psi + D_{r\nu(s)} \psi \end{array} .$$

The operator $\mathfrak{D}_r$ defines a Fredholm operator from $L^2_1(X; \underline{\mathbb{C}} \oplus K^{-1})$ to $L^2(X; \underline{\mathbb{C}} \oplus K^{-1})$ with certain boundary condition. According to [APS, theorem 3.10 and (4.3)], the index of $\mathfrak{D}_r$ is given by

$$\text{index}(\mathfrak{D}_r) = \frac{r^2}{32\pi^2} \int_Y a \wedge \mathrm{d}a + \frac{h_r + \eta_r}{2} - \frac{h_0 + \eta_0}{2} \tag{2.7}$$

where h_r is the dimension of $\ker D_r$ and η_r is the spectral asymmetry function of D_r. The spectral asymmetry function of a Dirac operator is defined as follows: It is the value at 0 of the analytic continuation to $\mathbb{C}$ of the function on the $u \gg 1$ part of the real line that sends u to

$$\sum_{\lambda \neq 0} \text{sign}(\lambda) |\lambda|^{-u}$$

where λ runs over the eigenvalues of the Dirac operator. Theorem 3.10 in [APS] asserts that this analytic continuation is finite at $u = 0$.

As noted in [APS], the index of $\mathfrak{D}_r$ is precisely $\text{sf}_a(r)$. Thus, (2.7), theorem 2.1 and proposition 3.1 (in next section) have the following to say about η_r.

Corollary 2.2. *With the same setting as theorem 2.1, there exists a constant c such that*

$$|\eta_r| \leq cr \qquad \textit{and} \qquad h_r \leq cr$$

for all $r \geq 1$.

Proof. Proposition 3.1 finds a constant c_1 such that there are only positive zero crossings for the spectral flow when $r \geq c_1$. Due to theorem 2.1, there exists a constant c_2 such that

$$h_r \leq \text{sf}_a(r+1) - \text{sf}_a(r-1) \leq c_2 r$$

for all $r \geq c_1 + 1$.

Since $\text{sf}_a(r) = \text{index}(\mathfrak{D}_r)$, (2.7) and theorem 2.1 implies that

$$|\eta_r| \leq h_r + h_0 + |\eta_0| + c_3 r$$

for some constant c_3. With the estimate on h_r, we conclude that $|\eta_r| \leq c_4 r$ for some constant c_4. $\square$

Remark. Throughout this paper, the constants c's only depend on the contact form and the adapted Riemannian metric, and do *not* depend on r. Within each proof, the subscription of the constants $c_{(*)}$'s is only for indicating that they might change (usually increase) after each step. The constants $c_{(*)}$'s in two different proof have nothing to do with each other.

3. Some estimates

The purpose of this section is to derive some basic estimates for the zero modes of D_r. As long as the Riemannian metric is adapted, the results in this section still hold. The argument works not only for the canonical $\mathrm{Spin}^{\mathbb{C}}$ bundle, but also for any $\mathrm{Spin}^{\mathbb{C}}$ bundle. Meanwhile, it has nothing to do with the open book decompositions.

For a section ψ of the canonical $\mathrm{Spin}^{\mathbb{C}}$ bundle, $\underline{\mathbb{C}} \oplus K^{-1}$, we write ψ as (α, β). They will be referred as the first and the second component, respectively. With an adapted metric, a $\mathrm{Spin}^{\mathbb{C}}$ bundle also split as a direct sum of two line bundles according to the Clifford multiplication of a. The Clifford action of a acts as i on the first summand, and acts as $-i$ on the second summand.

Proposition 3.1. *For any $\delta_1 \geq 0$, there is a constant c which is determined by the contact form and the adapted Riemannian metric, and which has the following significance:*

(i) *Suppose that ψ is a eigensection of D_r for some $r \geq c$, and the magnitude of the corresponding eigenvalue is less then or equal to δ_1. Then*

$$\int_Y |\beta|^2 + r^{-1} \int_Y |\nabla_r \beta|^2 \leq c r^{-1} \int_Y |\alpha|^2.$$

Hence, $\int_Y |D_r \beta|^2 \leq c \int_Y |\alpha|^2$.

(ii) *Furthermore, suppose that there is a $\mathcal{C}^1$-family of eigensections and eigenvalues, $D_{r+s}\psi(s) = \lambda(s)\psi(s)$ near some $r \geq c$, and $|\lambda(0)| \leq \delta_1$. Then*

$$|\lambda'(0) - \frac{1}{2}| \leq \frac{c}{r}.$$

Therefore, there exist only positive zero crossings for the spectral flow for all $r \geq c$.

Proof. The Weitzenböck formula reads

$$D_r^2 \psi = \nabla_r^* \nabla_r \psi + \frac{\kappa}{4}\psi + \mathrm{cl}(\frac{F_{A_0}}{2})\psi - ir\mathrm{cl}(*a)\psi$$

where κ is the scalar curvature and F_{A_0} is the curvature of the canonical connection. Take the inner product with β, and integrate over Y. After some simple manipulations, we have

$$\begin{aligned} \delta_1^2 \int_Y |\beta|^2 > \int_Y & |\nabla_r \beta|^2 + \langle N(\nabla_r \alpha) + N'(\alpha), \beta \rangle \\ & + \frac{\kappa}{4}|\beta|^2 + \mathrm{cl}(\frac{F_{A_0}}{2})|\psi|^2 + r|\beta|^2 \\ \geq \int_Y & (r - c_1)|\beta|^2 + \frac{1}{2}|\nabla_r \beta|^2 - c_2|\alpha|^2 \end{aligned}$$

where N and N' are defined from the covariant derivative of a, and depend only on the contact form and the Riemannian metric. Property (i) of the proposition follows from the last inequality.

Denote the differentiation of $\psi(s)$ in s by $\psi'(s)$. We can assume that $\psi(s)$ is of unit length for all s, then $\int_Y \langle\psi(s), \psi'(s)\rangle + \langle\psi'(s), \psi(s)\rangle = 0$. The differentiation of the Dirac equation in s is

$$-\frac{i}{2}\mathrm{cl}(a)\psi(s) + D_{r+s}\psi'(s) = \lambda'(s)\psi(s) + \lambda(s)\psi'(s).$$

At $s = 0$, take the inner product with $\psi(0)$, and integrate over Y. Since D_r is self-adjoint and $\psi(0)$ is an eigensection of D_r, we have

$$\lambda'(0) = \frac{1}{2}\int_Y |\alpha(0)|^2 - |\beta(0)|^2 = \frac{1}{2} - \int_Y |\beta(0)|^2.$$

This equation together with property (i) proves property (ii). □

Property (ii) of proposition 3.1 says that the rate of change of the spectral flow gets closer to $1/2$ as $r \to \infty$. It implies the following corollary.

Corollary 3.2. *For any $\delta_1 > 0$, there exist a constant $c > 0$ which is determined by the contact form and the adapted Riemannian metric, and which has the following significance: Suppose that ψ is an eigensection of D_r with eigenvalue λ, with $r \geq c$ and $4|\lambda| \leq \delta_1$. Then, ψ and λ associate with a continuous family of eigenvalues which contributes to the spectral flow with $+1$ somewhere in the interval*

$$[r - 2\lambda - \frac{c}{r}, r - 2\lambda + \frac{c}{r}].$$

For any two zero modes at different r, they might not be orthogonal to each other. However, property (i) of proposition 3.1 implies that they are not too far from being orthogonal. More precisely, we have:

Proposition 3.3. *There exists a constant $c > 0$ which is determined by the contact form and the adapted Riemannian metric, and which has the following significance: Suppose that ψ_1 and ψ_2 are zero modes of D_{r_1} and D_{r_2} with $r_1 \geq c$, $r_2 \geq c$ and $r_1 \neq r_2$, and they are of unit L^2-norm. Then*

$$|\int_Y \langle\psi_1, \psi_2\rangle| \leq c(r_1 r_2)^{-\frac{1}{2}}.$$

Furthermore, when $r_1 = r_2 \geq c$, the inequality still holds if $\int_Y \langle\psi_1, \psi_2\rangle = 0$.

Proof. When $r_1 \neq r_2$,

$$\begin{aligned} 0 &= \int_Y \langle D_{r_1}\psi_1, \psi_2\rangle - \langle\psi_2, D_{r_2}\psi_2\rangle = \int_Y \langle (D_{r_1} - D_{r_2})\psi_1, \psi_2\rangle \\ &= (r_1 - r_2)\int_Y \langle\alpha_1, \alpha_2\rangle - \langle\beta_1, \beta_2\rangle. \end{aligned}$$

If $r_1 = r_2$, the orthogonality says that $\int_Y \langle\alpha_1, \alpha_2\rangle + \langle\beta_1, \beta_2\rangle = 0$. By property (i) of proposition 3.1 and the Cauchy–Schwarz inequality, the inequality follows. □

4. The Dirac equations

We are going to write down the Dirac equations on different regions of the 3-manifold. The adpated Riemannian metric will also be specified.

4.1. **The tubular neighborhood of the binding.** For the tubular neighborhood of the binding, it has coordinate $(e^{it}, \rho e^{i\phi}) \in S^1 \times D^2$. The contact form is

$$a = f\mathrm{d}\phi + g\mathrm{d}t.$$

The contact form with the following two coframes

$$\begin{aligned}\omega^1 &= \cos\phi\mathrm{d}\rho - \sin\phi\big(\frac{f'}{2}\mathrm{d}\phi + \frac{g'}{2}\mathrm{d}t\big)\\ \omega^2 &= \sin\phi\mathrm{d}\rho + \cos\phi\big(\frac{f'}{2}\mathrm{d}\phi + \frac{g'}{2}\mathrm{d}t\big)\end{aligned}$$

determines the Riemannian metric. The Dirac operator is

$$\begin{aligned}&\frac{r}{2}\alpha + \frac{i}{2\Delta}(-g'\partial_\phi\alpha + f'\partial_t\alpha)\\ &\qquad + e^{-i\phi}\big(-\partial_\rho\beta + \frac{i}{\Delta}(g\partial_\phi\beta - f\partial_t\beta) + \frac{g-\Delta'}{\Delta}\beta\big),\\ &e^{i\phi}\big(\partial_\rho\alpha + \frac{i}{\Delta}(g\partial_\phi\alpha - f\partial_t\alpha)\big)\\ &\qquad - (\frac{r}{2} + 1 - \frac{g'}{2\Delta} + \frac{f''g' - f'g''}{8\Delta})\beta - \frac{i}{2\Delta}(-g'\partial_\phi\beta + f'\partial_t\beta)\end{aligned} \tag{4.1}$$

where $\Delta = \frac{1}{2}(f'g - fg')$. The volume form is $\Delta\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}t$. Note that the Dirac operator is invariant under the two S^1-actions in $e^{i\phi}$ and e^{it}.

At $\rho = 0$, the coordinate has a singularity, and we shall use the complex coordinate $z = \rho e^{i\phi}$. Remember that when $0 \le \rho \le 10\epsilon$, the function f is ρ^2, and g is $2 - \rho^2$. The Dirac operator is

$$\begin{aligned}&\frac{r}{2}\alpha + \frac{i}{2}(\partial_\phi\alpha + \partial_t\alpha) + (-2\partial_z\beta - \frac{i}{2}\bar{z}(\partial_\phi\beta + \partial_t\beta) - \frac{\bar{z}}{2}\beta),\\ &\qquad (2\partial_{\bar{z}}\alpha - \frac{i}{2}z(\partial_\phi\alpha + \partial_t\alpha)) - \frac{r+3}{2}\beta - \frac{i}{2}(\partial_\phi\beta + \partial_t\beta).\end{aligned} \tag{4.2}$$

The volume form is $2\rho\mathrm{d}\rho\wedge\mathrm{d}\phi\wedge\mathrm{d}t = i\mathrm{d}z\wedge\mathrm{d}\bar{z}\wedge\mathrm{d}t$, and we can regard (4.2) as an operator on $\mathbb{C}\times S^1$.

4.2. **The Dehn-twist region.** We will write down the equations on $(-1-15\epsilon, 1+15\epsilon)\times S^1\times[0,2\pi]$, and work with the operators and functions which are invariant under the identification map. Before taking the identification, we have coordinate $(\rho, e^{it}, \phi) \in (-1-15\epsilon, 1+15\epsilon)\times S^1\times[0,2\pi]$. With this understood, the contact form is

$$a = V\mathrm{d}\phi + 2(v-\rho)\mathrm{d}t + 2\phi(v-\rho)\tau'\mathrm{d}\rho.$$

The following two coframes

$$\omega^1 = \cos\phi \mathrm{d}\rho - \sin\phi(-\mathrm{d}t - \phi\tau'\mathrm{d}\rho - (v-\rho)\tau'\mathrm{d}\phi),$$
$$\omega^2 = \sin\phi \mathrm{d}\rho + \cos\phi(-\mathrm{d}t - \phi\tau'\mathrm{d}\rho - (v-\rho)\tau'\mathrm{d}\phi)$$

together with $\omega^3 = a$ give the Riemannian metric.

The Dirac operator is

$$\begin{aligned}
&\frac{r}{2}\alpha + \frac{i}{\tilde{\Delta}}\big(\partial_\phi\alpha - (v-\rho)\tau'\partial_t\alpha\big) + e^{-i\phi}\frac{2(v-\rho)-\tilde{\Delta}'}{\tilde{\Delta}}\beta \\
&\qquad + e^{-i\phi}\Big(-(\partial_\rho\beta - \phi\tau'\partial_t\beta) + \frac{iV}{\tilde{\Delta}}(-\partial_t\beta + \frac{2(v-\rho)}{V}\partial_\phi\beta)\Big), \\
&e^{i\phi}\Big((\partial_\rho\alpha - \phi\tau'\partial_t\alpha) + \frac{iV}{\tilde{\Delta}}(-\partial_t\alpha + \frac{2(v-\rho)}{V}\partial_\phi\alpha)\Big) \\
&\qquad - (\frac{r}{2} + 1 + \frac{1}{\tilde{\Delta}} + \frac{((v-\rho)\tau)''}{2\tilde{\Delta}})\beta - \frac{i}{\tilde{\Delta}}(\partial_\phi\beta - (v-\rho)\tau'\partial_t\beta)
\end{aligned}$$

where $\tilde{\Delta} = V - 2(v-\rho)^2\tau'$, and the volume form is $\tilde{\Delta}\mathrm{d}\phi\wedge\mathrm{d}t\wedge\mathrm{d}\rho$. Consider the *untwisting* of α and β:

$$\tilde{\alpha}(\rho,t,\phi) = \alpha(\rho, t-\phi\tau(\rho),\phi), \qquad \tilde{\beta}(\rho,t,\phi) = \beta(\rho,t-\phi\tau(\rho),\phi). \tag{4.3}$$

The Dirac operator becomes

$$\begin{aligned}
&\frac{r}{2}\tilde{\alpha} + \frac{i}{\tilde{\Delta}}\big(\partial_\phi\tilde{\alpha} - ((v-\rho)\tau)'\partial_t\tilde{\alpha}\big) + e^{-i\phi}\frac{2(v-\rho)-\tilde{\Delta}'}{\tilde{\Delta}}\tilde{\beta} \\
&\qquad + e^{-i\phi}\Big(-\partial_\rho\tilde{\beta} + \frac{i}{\tilde{\Delta}}((-V+2(v-\rho)\tau)\partial_t\tilde{\beta} + 2(v-\rho)\partial_\phi\tilde{\beta})\Big), \\
&e^{i\phi}\Big(\partial_\rho\tilde{\alpha} + \frac{i}{\tilde{\Delta}}((-V+2(v-\rho)\tau)\partial_t\tilde{\alpha} + 2(v-\rho)\partial_\phi\tilde{\alpha})\Big) \\
&\qquad - (\frac{r}{2} + 1 + \frac{1}{\tilde{\Delta}} + \frac{((v-\rho)\tau)''}{2\tilde{\Delta}})\tilde{\beta} - \frac{i}{\tilde{\Delta}}(\partial_\phi\tilde{\beta} - ((v-\rho)\tau)'\partial_t\tilde{\beta}).
\end{aligned} \tag{4.4}$$

The untwisting operator (4.3) and the Dirac equation (4.4) have the following features:

(i) The untwisting operator (4.3) is only defined locally. In general, it cannot be extended to the whole 3-manifold;

(ii) Note that $\tilde{\alpha}(\rho,t,2\pi) = \alpha(\rho,t-2\pi\tau(\rho),2\pi) = \alpha(\rho,t,0) = \tilde{\alpha}(\rho,t,0)$. Hence, after the untwisting, $\tilde{\alpha}$ and $\tilde{\beta}$ are 2π-periodic in both t and ϕ, and the Dirac operator (4.4) is also invariant under these two S^1-actions;

(iii) The Dirac operator (4.4) is of the same form as that on the tubular neighborhood of the binding (4.1). More precisely, it corresponds to $\tilde{f} = V - 2(v-\rho)\tau$ and $\tilde{g} = 2(v-\rho)$ in (4.1), and $\tilde{\Delta}$ is equal to $\frac{1}{2}(\tilde{f}'\tilde{g} - \tilde{f}\tilde{g}')$.

4.3. **Associated contact form on $S^2 \times S^1$.** It is convenient to think the Dirac operators in subsections 4.1 and 4.2 as defined on $S^2 \times S^1$.

We now describe how $S^2 \times S^1$ arise from our 3-manifold Y. Let $(\rho e^{it}, e^{i\phi})$ be the coordinate on an Annulus $\times S^1$, which can be viewed as the mapping torus of the Annulus with the identity map. With the construction of the open book decomposition, the resulting 3-manifold is $S^2 \times S^1$. The coordinate e^{it} becomes the coordinate on the S^1-component. The coordinates ρ and ϕ are the (reparametrized) azimuthal angle and the equatorial angle on the S^2-component. With this understood, consider contact forms of the following type

$$a = f(\rho)\mathrm{d}\phi + g(\rho)\mathrm{d}t \tag{4.5}$$

on $S^2 \times S^1$. For such 1-forms, $a \wedge \mathrm{d}a = 2\Delta \mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}t$ where

$$\Delta = \frac{1}{2}(f'g - fg'). \tag{4.6}$$

There are some conditions on f and g in order for (4.5) to be a genuine contact form. We now describe them in our case.

On the tubular neighborhood of the binding. The contact form is of the type (4.5) for $\rho \in [0, 1+10\epsilon]$. When $\rho \in [1-5\epsilon, 1+15\epsilon]$, the function f is V, and g is $2-\rho$. Extend f and g to $\rho \leq 2$ such that

- When $\rho \in [2-10\epsilon, 2]$, the function f is $(2-\rho)^2$, and g is $-2+(2-\rho)^2$;
- The functions f and Δ are always positive, except at $\rho = 2$.

It is clear that such extension always exists. After the extension, $f\mathrm{d}\phi + g\mathrm{d}t$ defines a contact form on $S^2 \times S^1$, with $\rho \in [0, 2]$.

On the Dehn-twist region. As observed in subsection 4.2, the Dirac operator after untwisting is the same as that on the tubular neighborhood of the binding, with $\tilde{f} = V - 2(v-\rho)\tau$ and $\tilde{g} = 2(v-\rho)$ for $\rho \in [-1-15\epsilon, 1+15\epsilon]$. Extend them to $-2 \leq \rho \leq 2$ in a similar way:

- When $\rho \in [2-10\epsilon, 2]$, the function $\tilde{f}$ is $(2-\rho)^2$, and $\tilde{g}$ is $-2|v| - 2 + (2-\rho)^2$;
- When $\rho \in [-2, -2+10\epsilon]$, the function $\tilde{f}$ is $(\rho+2)^2$, and $\tilde{g}$ is $2|v| + 2 - (\rho+2)^2$;
- The functions $\tilde{f}$ and $\tilde{\Delta}$ are always positive, except at $\rho = \pm 2$.

Then $\tilde{f}\mathrm{d}\phi + \tilde{g}\mathrm{d}t$ also defines a contact form on $S^2 \times S^1$, with $\rho \in [-2, 2]$.

Definition 4.1. For each boundary component and Dehn-twist region of the page, the above extension gives a contact form (4.5) on $S^2 \times S^1$. These contact forms will be called the *associated contact forms.*

We will study the Dirac equations of the associated contact forms carefully in section 5.

4.4. **The part with trivial monodromy.** On the part of the page where the monodromy is the identity map, the contact form is

$$a = V\mathrm{d}\phi + 2\mu_\Sigma.$$

To start, choose a Riemannian metric $\mathrm{d}s^2_\Sigma$ on Σ such that

- The area form is $\mathrm{d}\mu_\Sigma$;
- Near the tubular neighborhood of the binding, $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \frac{1}{4}\mathrm{d}t^2$ in terms of the coordinates in subsection 4.1;
- Near the Dehn-twist region, $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{d}t^2$ in terms of the coordinates in subsection 4.2.

Any local oriented orthonormal frame on Σ, u_1 and u_2, gives rise to the following frame for the contact hyperplane on the 3-manifold:

$$\begin{aligned} e_1 &= \cos\phi u_1 - \sin\phi u_2 - \frac{2}{V}\mu_\Sigma(\cos\phi u_1 - \sin\phi v_2)\partial_\phi, \\ e_2 &= \sin\phi u_1 + \cos\phi u_2 - \frac{2}{V}\mu_\Sigma(\sin\phi u_1 + \cos\phi v_2)\partial_\phi. \end{aligned}$$

The Dirac operator is

$$\begin{aligned} &\frac{r}{2}\alpha + \frac{i}{V}\partial_\phi\alpha + e^{-i\phi}\big(- u_1(\beta) + iu_2(\beta) - \frac{2}{V}\mu_\Sigma(-u_1 + iu_2)\partial_\phi\beta \\ &\qquad\qquad\qquad + \frac{2i}{V}\mu_\Sigma(-u_1 + iu_2)\beta + i\theta^2_1(-u_1 + iu_2)\beta\big), \\ &e^{i\phi}\big(u_1(\alpha) + iu_2(\alpha) - \frac{2}{V}\mu_\Sigma(u_1 + iu_2)\partial_\phi\alpha\big) - (\frac{r}{2} + 1 + \frac{1}{V})\beta - \frac{i}{V}\partial_\phi\beta \end{aligned}$$

where θ^2_1 is the Levi-Civita connection of $\mathrm{d}s^2_\Sigma$.

Consider the separation of variables:

$$\alpha = \alpha_n e^{in\phi}(2\pi V)^{-\frac{1}{2}}, \qquad \beta = \beta_n e^{i(n+1)\phi}(2\pi V)^{-\frac{1}{2}}. \tag{4.7}$$

The Dirac operator on the frequency n components is

$$\begin{aligned} &(\frac{r}{2} - \frac{n}{V})\alpha_n + \bar\partial^*_n\beta_n, \\ &\bar\partial_n\alpha_n - (\frac{r}{2} + 1 - \frac{n}{V})\beta_n \end{aligned} \tag{4.8}$$

where $\bar\partial_n$ and $\bar\partial^*_n$ are the Cauchy–Riemann operators on $\underline{\mathbb{C}} \oplus K^{-1}_\Sigma$ with the connection perturbed by $-\frac{2in}{V}\mu_\Sigma$. Their index is given by [APS, theorem 3.10 and (4.3)], and the formula reads

$$\dim\ker\bar\partial_n - \dim\ker\bar\partial^*_n = \frac{n}{V\pi}\iint_\Sigma \mathrm{d}\mu_\Sigma + \frac{1}{2}\chi(\Sigma) + \frac{1}{2}(\eta_n + h_n)(\partial\Sigma) \tag{4.9}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, and $\eta_n(\partial\Sigma)$ and $h_n(\partial\Sigma)$ are the correction terms from the boundary. Since the boundary of Σ is a disjoint union of *circles*, the correction terms, $\eta_n(\partial\Sigma)$ and $h_n(\partial\Sigma)$, are uniformly bounded for all n.

In [APS], the connection is required to depend only on $\partial\Sigma$ in a small neighborhood of $\partial\Sigma$. With the notation in subsections 4.1 and 4.2, the

connection is required to be independent of ρ near $\partial\Sigma$. Our connection μ_Σ does not satisfy this property. However, μ_Σ is affine in ρ, and $\partial\Sigma$ is a disjoint union of S^1's. With a slight modification of the proof, the index formula (4.9) still holds in our setting.

In [APS], there are adjoint boundary conditions for $\bar\partial_n$ and $\bar\partial_n^*$: The restriction of $\bar\partial_n$ or $\bar\partial_n^*$ on the boundary, without the part of taking derivative along the transverse direction ($\pm\partial_\rho$), is a Dirac operator on the boundary. Denote the operator by $\not{\partial}_n$. The Atiyah–Patodi–Singer boundary condition says that the restriction of α_n on the boundary only has components in the negative eigenspaces of $\not{\partial}_n$, and β_n only has non-negative ones. We now give an explicit description of the boundary conditions.

Adjacent to the tubular neighborhood of the binding. We follow the notations in subsection 4.1. The surface Σ is described by $\rho \geq 1$. The operators $\bar\partial_n$ and $\bar\partial_n^*$ are

$$\begin{aligned} \bar\partial_n\alpha_n &= \partial_\rho\alpha_n - 2i\partial_t\alpha_n - \frac{2n(2-\rho)}{V}\alpha_n, \\ \bar\partial_n^*\beta_n &= -\partial_\rho\beta_n - 2i\partial_t\beta_n - \frac{2n(2-\rho)}{V}\beta_n, \end{aligned} \tag{4.10}$$

and $\not{\partial}_n$ is $-2i\partial_t - \frac{2n}{V}$. Let $\alpha_n = \alpha_{n,m}(\rho)e^{imt}$ and $\beta_n = \beta_{n,m}(\rho)e^{imt}$. The Atiyah–Patodi–Singer boundary conditions are

$$\begin{aligned} \alpha_{n,m}(1) &= 0 \qquad \text{when } m \geq \frac{n}{V}, \\ \beta_{n,m}(1) &= 0 \qquad \text{when } m < \frac{n}{V}. \end{aligned} \tag{4.11}$$

Adjacent to the Dehn-twist region. We follow the notations in subsection 4.2. The region Σ is the union of where $\rho \geq 1$ and $\rho \leq -1$. The operators $\bar\partial_n$ and $\bar\partial_n^*$ are

$$\begin{aligned} \bar\partial_n\alpha_n &= \partial_\rho\alpha_n - i\partial_t\alpha_n - \frac{2n(v-\rho)}{V}\alpha_n, \\ \bar\partial_n^*\beta_n &= -\partial_\rho\beta_n - i\partial_t\beta_n - \frac{2n(v-\rho)}{V}\beta_n. \end{aligned} \tag{4.12}$$

At $\rho = 1$, $\not{\partial}_n$ is $-i\partial_t - \frac{2n(v-1)}{V}$. At $\rho = -1$, since ∂_ρ does not point inward, $\not{\partial}_n$ is $i\partial_t + \frac{2n(v+1)}{V}$. Let $\alpha_n = \alpha_{n,m}(\rho)e^{imt}$ and $\beta_n = \beta_{n,m}(\rho)e^{imt}$. The Atiyah–Patodi–Singer boundary conditions at $\rho = 1$ are

$$\begin{aligned} \alpha_{n,m}(1) &= 0 \qquad \text{when } m \geq 2n\frac{v-1}{V}, \\ \beta_{n,m}(1) &= 0 \qquad \text{when } m < 2n\frac{v-1}{V}. \end{aligned} \tag{4.13}$$

The conditions at $\rho = -1$ are

$$
(4.14) \qquad \begin{aligned} \alpha_{n,m}(-1) &= 0 \qquad \text{when } m \leq 2n\frac{v+1}{V}, \\ \beta_{n,m}(-1) &= 0 \qquad \text{when } m > 2n\frac{v+1}{V}. \end{aligned}
$$

In order to use the index formula (4.9) to do the counting, we need to know that $\dim \ker \bar{\partial}_n^* = 0$.

Lemma 4.2. *There exists a constant c which has the following significance: For all $n \geq c$, if β_n satisfies the Atiyah–Patodi–Singer boundary condition,*

$$
\int_\Sigma |\beta_n|^2 \leq cn^{-1} \int_\Sigma |\bar{\partial}_n^* \beta_n|^2.
$$

In particular, $\bar{\partial}_n^*$ only has trivial solution for any $n \geq c$.

Proof. The integration by parts formula gives

$$
\int_\Sigma |\bar{\partial}_n^* \beta_n|^2 = \int_\Sigma |\nabla_n \beta_n|^2 + \int_\Sigma (\frac{2n}{V} + \frac{\kappa_\Sigma}{4})|\beta_n|^2 + \int_{\partial\Sigma} \langle \not{\partial}_n \beta_n, \beta_n \rangle
$$

where κ_Σ is the scalar curvature. The Atiyah–Patodi–Singer boundary condition for β_n implies that $\int_{\partial\Sigma} \langle \not{\partial}_n \beta_n, \beta_n \rangle$ is non-negative. Hence, if $n \geq V \max |\kappa_\Sigma|$, we obtain the inequality claimed by the lemma. $\square$

For any integer $n \geq c$ in lemma 4.2, the dimension of $\ker \bar{\partial}_n$ is given by the right hand side of (4.9). Solutions of $\bar{\partial}_n$ automatically solve the Dirac equation (4.8) with r to be

$$
(4.15) \qquad \gamma_n = \frac{2n}{V}.
$$

However, they only solve the Dirac equation on $\Sigma \times S^1$. In order to get smooth sections on the 3-manifold Y, we need to do some modifications.

Definition 4.3. For any $\delta \in (-15\epsilon, 5\epsilon)$, let Σ_δ be the extension/curtailment of Σ defined by

- $\{\rho \geq 1 - \delta\}$ for the part adjacent to the tubular neighborhood of the binding, in terms of the coordinate in subsection 4.1;
- $\{\rho \leq -1 + \delta \text{ or } \rho \geq 1 - \delta\}$ for the part adjacent to the Dehn-twist region, in terms of the coordinate in subsection 4.2.

Positive δ corresponds to the extension, and negative δ corresponds to the curtailment. When $\delta = 0$, $\Sigma_0 = \Sigma$.

Let χ_Σ be the cut-off function which is equal to 1 on $\Sigma_\epsilon \times S^1$ and equal to 0 on $Y \backslash (\Sigma_{2\epsilon} \times S^1)$, and only depends on ρ over $(\Sigma_{2\epsilon} \backslash \Sigma_\epsilon) \times S^1$ in terms of the coordinate in subsections 4.1 and 4.2,

Suppose that α_n solves $\bar{\partial}_n$. On the part adjacent to the tubular neighborhhod of the binding, α_n is equal to

$$\sum_{m<\frac{n}{V}} \mathfrak{c}_{n,m} \exp\big(-\frac{n}{V}(\rho - 2 + \frac{Vm}{n})^2\big)e^{imt} \tag{4.16}$$

where $\mathfrak{c}_{n,m}$ are constants. The expression also solves $\bar{\partial}_n$ on where $\rho \geq 1-2\epsilon$, and it also satisfies the corresponding Atiyah–Patodi–Singer boundary condition. On the part adjacent to the Dehn-twist region, the situation is similar. Therefore, any solution of $\bar{\partial}_n$ on Σ can be extended uniquely to a solution on $\Sigma_{2\epsilon}$, and the extension satisfies the corresponding Atiyah–Patodi–Singer boundary condition on $\Sigma_{2\epsilon}$.

Consider the following construction of the almost eigensections: for each solution of $\bar{\partial}_n$, extend it to $\Sigma_{2\epsilon}$, and multiply it by the cut-off function χ_Σ. This process is linear, and it ends up with a vector space of the same dimension as $\ker \bar{\partial}_n$. Choose an orthonormal basis with respect to the L^2-inner product on $\Sigma_{2\epsilon}$. Denote the basis by $\{\xi_{n,l}\}$, where l runs from 1 to the number on the right hand side of (4.9). They are smooth functions on Y, and their properties are summarized in the following proposition.

Proposition 4.4. *There exists a constant c which has the following significance: For any integer $n \geq c$, let $\psi_{n,l}$ be the section whose first component is*

$$\xi_{n,l}e^{in\phi}(2\pi V)^{-\frac{1}{2}}$$

and second component is zero, where $\xi_{n,l}$ is given by the above construction. Then,

$$\int_Y |D_r\psi_{n,l} - \frac{r-\gamma_n}{2}\psi_{n,l}|^2 \leq c\exp(-\frac{n}{c})$$

for any $r > 0$, and γ_n is defined in (4.15). Also,

$$\int_Y \langle \psi_{n,l}, \psi_{n,l'}\rangle = \int_Y \langle D_r\psi_{n,l}, \psi_{n,l'}\rangle = 0,$$
$$\Big|\int_Y \langle D_r\psi_{n,l}, D_r\psi_{n,l'}\rangle\Big| \leq c\exp(-\frac{n}{c})$$

for any $l \neq l'$.

Proof. For each section $\psi_{n,l}$, there exists a function $\alpha_{n,l}$ which solves $\bar{\partial}_n$ on $\Sigma_{2\epsilon}$ and is extended from Σ, such that $\xi_{n,l} = \chi_\Sigma\alpha_{n,l}$. From the expression (4.16), there exists a constant c_1 such that

$$\int_{\Sigma_{2\epsilon}\setminus\Sigma_\epsilon} |\alpha_{n,l}|^2 \leq c_1\exp(-\frac{n}{c_1})\int_{\Sigma_\epsilon} |\alpha_{n,l}|^2. \tag{4.17}$$

By (4.8), the first component of $D_r\psi_{n,l}$ is

$$\frac{r-\gamma_n}{2}\chi_\Sigma\alpha_{n,l}e^{in\phi}(2\pi V)^{-\frac{1}{2}}.$$

The second component of $D_r\psi_{n,l}$ only supports on $\Sigma_{2\epsilon}\backslash\Sigma_\epsilon$, and is equal to

$$\chi'_\Sigma \alpha_{n,l} e^{i(n+1)\phi}(2\pi V)^{-\frac{1}{2}}.$$

Since $\{\chi_\Sigma \alpha_{n,l}\}$ forms an orthonormal set in $L^2(\Sigma_{2\epsilon})$, the above expression of $D_r\psi_{n,l}$ with (4.17) proves the proposition. $\square$

5. The model case: $S^2\times S^1$

In this section, we discuss the Dirac equations of associated contact forms (4.5) on $S^2\times S^1$. To distinguish from the original 3-manifold Y, the $S^2\times S^1$ of the tubular neighborhood of the binding will be denoted by $\check{Y}$. The Dirac operators will be denoted by $\check{D}_r$, and the sections will be denoted by $\check{\psi}$. The $S^2\times S^1$ of the tubular neighborhood of the binding will be denoted by $\tilde{Y}$. The Dirac operators will be denoted by $\tilde{D}_r$, and the sections will be denoted by $\tilde{\psi}$. Their spectral flow function will be denoted by $\check{\mathrm{sf}}_a(r)$ and $\tilde{\mathrm{sf}}_a(r)$, respectively. We will focus on the associated contact form of the tubular neighborhood of the binding. For the associated contact form of the Dehn-twist region, the argument is the same up to some changes of constants, and the detail will be omitted.

The coframes in subsection 4.1 are globally defined on $S^2\times S^1$, and they also trivialize $\underline{\mathbb{C}}\oplus K^{-1}$. The Dirac operator is still given by (4.1), and it is invariant under the two *global* S^1-actions in $e^{i\phi}$ and e^{it}. Thus, the eigenspaces of the Dirac operator split according to the frequencies with respect to these two S^1-actions. The splitting allows us to study the spectral flow function directly. Let $\mathcal{S}_{k,m}$ be the space of sections whose first component have frequency k in $e^{i\phi}$ and m in e^{it}, and whose second component have frequency $k+1$ in $e^{i\phi}$ and m in e^{it}, with the above global trivialization.

The following notions will be used throughout the paper.

Definition 5.1. For the associated contact form (4.5) of the tubular neighborhood of the binding, the function g/f is monotone decreasing in ρ. For each positive integer k and integer m, there is a unique $\check{\rho}_{k,m}\in(0,2)$ such that $kg(\check{\rho}_{k,m})=mf(\check{\rho}_{k,m})$. Let $\check{\gamma}_{k,m}$ to be

$$\check{\gamma}_{k,m}=\frac{mf'(\check{\rho}_{k,m})-kg'(\check{\rho}_{k,m})}{\Delta(\check{\rho}_{k,m})}=\frac{2k}{f(\check{\rho}_{k,m})}=\frac{2m}{g(\check{\rho}_{k,m})}$$

where Δ is defined by (4.6). The last equality only makes sense at where $g(\check{\rho}_{k,m})\neq 0$. If $k=0$ and $m>0$, let $\check{\rho}_{k,m}$ be 0, and $\check{\gamma}_{k,m}$ be m. If $k=0$ and $m<0$, let $\check{\rho}_{k,m}$ be 2, and $\check{\gamma}_{k,m}$ be $-m$.

For the associated contact form of the Dehn-twist region, $\tilde{\gamma}_{k,m}$ is defined in the same way: Just replace f, g and Δ by $\tilde{f}$, $\tilde{g}$ and $\tilde{\Delta}$, and $\rho_{k,m}$ lies in the interval $[-2,2]$. For $k=0$ and $m\neq 0$, $\tilde{\gamma}_{k,m}=\frac{\mathrm{sign}(m)m}{|v|+1}$.

We will have various cut-off functions for different purpose. They will be denoted by χ with some sub-/super-scription. If there is no sub-/super-scription, it is the following one:

Definition 5.2. Let $\chi(x)$ be the cut-off function on $\mathbb{R}$ with $\chi(x) = 1$ when $|x| \leq \frac{1}{2}$ and $\chi(x) = 0$ when $|x| \geq 1$.

5.1. **Uniqueness of zero crossing.** In order to prove the upper bound in theorem 2.1 for the associated contact forms, we need to know that the zero crossing of $\check{D}_r$ on each $\mathcal{S}_{k,m}$ is unique.

Proposition 5.3. *For the associated contact form (4.5) on $\check{Y}$, there exists a constant $c > 0$ which has the following significance:*

(i) *For each k and m, the Dirac operator $\check{D}_r$ on $\mathcal{S}_{k,m}$ has at most one zero crossing for $r \geq c$.*
(ii) *If k is negative, or both k and m are zero, there is no zero crossing for $r \geq c$.*
(iii) *If the Dirac operator does have a zero crossing on $\mathcal{S}_{k,m}$ at some $r \geq c$, then*
$$r \in [\check{\gamma}_{k,m} - c, \check{\gamma}_{k,m} + c].$$

Proof. Suppose that $\check{\psi} = (\check{\alpha}, \check{\beta}) \in \mathcal{S}_{k,m}$ is a zero mode at r. Proposition 3.1 finds a constant c_1 such that

$$\int_{\check{Y}} \big|(r + \frac{kg' - mf'}{\Delta})\check{\alpha}\big|^2 \leq c_1 \int_{\check{Y}} |\check{\alpha}|^2, \tag{5.1}$$

$$\int_{\check{Y}} \big|e^{i\phi}(\partial_\rho \check{\alpha} - \frac{kg - mf}{\Delta}\check{\alpha})\big|^2 \leq c_1 \int_{\check{Y}} |\check{\alpha}|^2 \tag{5.2}$$

provided $r \geq c_1$. Denote $(r + \frac{kg'-mf'}{\Delta})\check{\alpha}$ by $\mathcal{D}_1\check{\alpha}$, and $e^{i\phi}(\partial_\rho\check{\alpha} - \frac{kg-mf}{\Delta}\check{\alpha})$ by $\mathcal{D}_2\check{\alpha}$. They are the dominant terms of the Dirac equation.

Rough relations between k, m and r. Consider the integral of $\langle \mathcal{D}_1\check{\alpha}, f\check{\alpha}\rangle + \langle \mathcal{D}_2\check{\alpha}, e^{i\phi} f'\check{\alpha}\rangle$ and the integral of $\langle \mathcal{D}_1\check{\alpha}, g\check{\alpha}\rangle + \langle \mathcal{D}_2\check{\alpha}, e^{i\phi} g'\check{\alpha}\rangle$. With (5.1), (5.2) and after some simple manipulations, there exist a constant c_2 such that

$$\begin{aligned} -c_2 \int_{\check{Y}} |\check{\alpha}|^2 &\leq \int_{\check{Y}} (rf - 2k)|\check{\alpha}|^2 \leq c_2 \int_{\check{Y}} |\check{\alpha}|^2, \\ -c_2 \int_{\check{Y}} |\check{\alpha}|^2 &\leq \int_{\check{Y}} (rg - 2m)|\check{\alpha}|^2 \leq c_2 \int_{\check{Y}} |\check{\alpha}|^2 \end{aligned} \tag{5.3}$$

provided $r \geq c_1$. Let $c_3 = c_2 + \max\{f, |g|\}$, then (5.3) implies that

$$2k \leq rc_3, \qquad\qquad 2|m| \leq rc_3 \tag{5.4}$$

provided $r \geq c_1$. The bound of r in terms of k and m is straightforward. By (5.1), there exists a constant c_4 such that

$$r \leq c_4(|k| + |m|) \tag{5.5}$$

provided $r \geq c_4$. It rules out the possibility that both k and m are zero.

Some estimates on α. Consider the integral of $|f\mathcal{D}_1\check{\alpha}+e^{-i\phi}f'\mathcal{D}_2\check{\alpha}|^2$, $|g\mathcal{D}_1\check{\alpha}+e^{-i\phi}g'\mathcal{D}_2\check{\alpha}|^2$, and $|e^{-i\phi}\Delta\mathcal{D}_2\check{\alpha}|^2$. With (5.1), (5.2) and after some simple manipulations, there exists a constant c_5 such that

$$\int_{\check{Y}}(rf-2k)^2|\check{\alpha}|^2 \leq rc_5\int_{\check{Y}}|\check{\alpha}|^2, \tag{5.6}$$

$$\int_{\check{Y}}(rg-2m)^2|\check{\alpha}|^2 \leq rc_5\int_{\check{Y}}|\check{\alpha}|^2, \tag{5.7}$$

$$\int_{\check{Y}}(kg-mf)^2|\check{\alpha}|^2 \leq rc_5\int_{\check{Y}}|\check{\alpha}|^2 \tag{5.8}$$

provided $r \geq c_1$.

We separate the discussion into two cases according to whether

$$|m| < (\frac{1}{32\epsilon^2}-1)k \qquad \text{or} \qquad |m| \geq (\frac{1}{32\epsilon^2}-1)k.$$

Case 1. When $|m| < (\frac{1}{32\epsilon^2}-1)k$, k can only be positive. We are going to use (5.8) to obtain a refined estimate on α. Note that $\check{\rho}_{n,m}$ given by definition 5.1 lies within $(8\epsilon, 2-8\epsilon)$. The function $|kg-mf|$ can only be small near $\check{\rho}_{n,m}$. More precisely, there is a constant $c_6>0$ such that

$$|ng-mf| \geq \begin{cases} \frac{1}{c_6}r & \text{when } |\rho-\check{\rho}_{k,m}| \geq \epsilon, \\ \frac{1}{c_6}r|\rho-\rho_{k,m}| & \text{when } |\rho-\check{\rho}_{k,m}| < \epsilon. \end{cases} \tag{5.9}$$

provided $r \geq c_6$. Here is its proof: The condition $|m| < (\frac{1}{32\epsilon^2}-1)k$ and (5.5) implies that k is greater than some multiple of r. When $\rho \leq 7\epsilon$ or $\rho \geq 2-7\epsilon$, it is straightforward to verify (5.9). When $\rho \in (7\epsilon, 2-7\epsilon)$, (5.9) follows from the Taylor's theorem on $\frac{1}{f}(kg-mf)$ and the monotonicity of $\frac{1}{f}(kg-mf)$.

With (5.8), and (5.9), there exist a constant c_7 such that

$$\int_{\check{Y}}|\check{\alpha}|^2 \leq c_7\int_{|\rho-\check{\rho}_{k,m}|\leq c_7r^{-\frac{1}{2}}}|\check{\alpha}|^2 \tag{5.10}$$

provided $r \geq c_7$.

Refined estimate on r. By (5.1) and (5.10),

$$\begin{aligned} c_1\int_{\check{Y}}|\check{\alpha}|^2 &\geq \int_{|\rho-\check{\rho}_{k,m}|\leq c_7r^{-\frac{1}{2}}}\big|(r-\frac{mf'-kg'}{\Delta})\check{\alpha}\big|^2 \\ &\geq \int_{|\rho-\check{\rho}_{k,m}|\leq c_7r^{-\frac{1}{2}}}\Big(\frac{(r-\check{\gamma}_{k,m})^2}{2}|\check{\alpha}|^2-\big|(\check{\gamma}_{k,m}-\frac{mf'-kg'}{\Delta})\check{\alpha}\big|^2\Big) \\ &\geq \int_{\check{Y}}\frac{(r-\check{\gamma}_{k,m})^2}{2c_7}|\check{\alpha}|^2-c_8\int_{\check{Y}}|\check{\alpha}|^2. \end{aligned}$$

For the last term in last inequality, note that the derivative of $\frac{mf'-kg'}{\Delta}$ at $\check{\rho}_{k,m}$ is zero. The Taylor's theorem on $\check{\gamma}_{k,m} - \frac{mf'-kg'}{\Delta}$ implies that

$$\big|\check{\gamma}_{k,m} - \frac{mf'-kg'}{\Delta})\big|^2 \leq c_8' r^2 |\rho - \check{\rho}_{k,m}|^4 \leq c_8$$

for any ρ with $|\rho - \check{\rho}_{k,m}| \leq c_7 r^{-\frac{1}{2}}$.

Thus, there exits a constant c_9 such that

$$|r - \check{\gamma}_{k,m}| \leq c_9 \tag{5.11}$$

provided $r \geq c_9$. It says that any zero crossing on $\mathcal{S}_{k,m}$ must happen somewhere very close to $\check{\gamma}_{k,m}$. It also allows us to replace $\check{\gamma}_{k,m}$ by r for doing the estimates.

Zeroth order approximation of $\check{\alpha}$. Consider the linearization of $e^{-i\phi}\mathcal{D}_2$ at $\check{\rho}_{k,m}$:

$$\mathcal{L}_{k,m} = \partial_x + \check{\gamma}_{k,m} x$$

where x is $\rho - \check{\rho}_{k,m}$. If we regard $\mathcal{L}_{k,m}$ as an operator on $\mathbb{R}$, the theory of the 1-dimensional harmonic oscillator applies. See [R, chapter 9] for the properties of the harmonic oscillator. If $\check{\gamma}_{k,m} > 0$, $\mathcal{L}_{k,m}$ has the following properties: Its kernel is 1-dimensional, and is spanned by

$$\check{\xi}_{k,m} = (\frac{\check{\gamma}_{k,m}}{\pi})^{\frac{1}{4}} \exp(-\frac{\check{\gamma}_{k,m}}{2} x^2) \tag{5.12}$$

It has a right inverse operator $G_{k,m} : \mathcal{C}^\infty_{\text{cpt}}(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ which satisfies

$$\int_{\mathbb{R}} \langle G_{k,m}\eta, \check{\xi}_{k,m}\rangle = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} |G_{k,m}\eta|^2 \leq \frac{1}{\check{\gamma}_{k,m}} \int_{\mathbb{R}} |\eta|^2$$

for any $\eta \in \mathcal{C}^\infty_{\text{cpt}}(\mathbb{R})$.

Consider the cut-off function $\chi(r^{\frac{1}{3}}x)$. By (5.8), (5.9), (5.2) and (5.11), there exists a constant c_{10} such that

(5.13)

$$\int_{\check{Y}} \Big|\big(1 - \chi(r^{\frac{1}{3}}x)\big)\check{\alpha}\Big|^2 \leq c_{10} r^{-\frac{1}{3}} \int_{\check{Y}} |\check{\alpha}|^2,$$

(5.14)

$$\begin{aligned}\int_{\check{Y}} \Big|\mathcal{L}_{k,m}\big(\chi(r^{\frac{1}{3}}x)\check{\alpha}\big)\Big|^2 &\leq c_{10}'\Big(\int \big|\partial_x(\chi(r^{\frac{1}{3}}x))\,\check{\alpha}|^2 + r^2 \int_{|x|\leq r^{-\frac{1}{3}}} x^4|\check{\alpha}|^2\Big) \\ &\leq c_{10}''\Big(r^{\frac{2}{3}} \int_{\frac{1}{2}r^{-\frac{1}{3}}\leq|x|\leq r^{-\frac{1}{3}}} |\check{\alpha}|^2 + r^{\frac{4}{3}} \int_{|x|\leq r^{-\frac{1}{3}}} x^2|\check{\alpha}|^2\Big) \\ &\leq c_{10} r^{\frac{1}{3}} \int_{\check{Y}} |\check{\alpha}|^2\end{aligned}$$

provided $r \geq c_{10}$. Let

$$\chi(r^{\frac{1}{3}}x)\check{\alpha} = \check{\alpha}_{k,m}(x) e^{i(k\phi+mt)} \Delta^{-\frac{1}{2}} (2\pi)^{-1}.$$

If we regard $\check{\alpha}_{k,m}(x)$ as defined on $\mathbb{R}$ and apply $G_{k,m}$ on $\mathcal{L}_{k,m}\big(\check{\alpha}_{k,m}(x)\big)$, we conclude that

$$\begin{aligned}(5.15)\qquad & \check{\alpha}_{k,m} = \mathfrak{c}_{k,m}\check{\xi}_{k,m} + \check{\xi}^{\perp}_{k,m} \\ & \text{with } |\mathfrak{c}_{k,m}|^2 \geq (1 - c_{11}r^{-\frac{1}{6}}) \int_{\check{Y}} |\check{\alpha}|^2, \\ & \text{and } \int_{\mathbb{R}} |\check{\xi}^{\perp}_{k,m}|^2 \leq c_{11}r^{-\frac{2}{3}} \int_{\check{Y}} |\check{\alpha}|^2\end{aligned}$$

for some constant c_{11}.

Now, (5.15), (5.13) and (5.11) find a constant c_{12} which have the following significance: Suppose that $\check{D}_r$ on $\mathcal{S}_{k,m}$ has two zero modes $\check{\psi}_1$ and $\check{\psi}_2$ at $r_1 \geq c_{12}$ and $r_2 \geq c_{12}$, respectively, then,

$$\Big|\int_{\check{Y}} \langle \check{\alpha}_1, \check{\alpha}_2\rangle\Big|^2 \geq (1 - c_{12}r^{-\frac{1}{6}}) \int_{\check{Y}} |\check{\alpha}_1|^2 \int_{\check{Y}} |\check{\alpha}_2|^2.$$

This contradicts proposition 3.3, and the uniqueness in case 1 follows.

Case 2 with $m > 0$. When $|m| \geq (\frac{1}{32\epsilon^2} - 1)k$, let us further assume that $m > 0$. For $m < 0$, it is essentially the same, and we will mention it briefly at the end of the proof. The first task is to show that $\check{\alpha}$ is small when $\rho \geq 9\epsilon$. To start, (5.7) implies that

$$\int_{\rho \geq 2 - 9\epsilon} |\check{\alpha}|^2 \leq c_5 r^{-1} \int_{\check{Y}} |\check{\alpha}|^2.$$

If $k \leq 0$, (5.6) implies that

$$\int_{9\epsilon \leq \rho \leq 2 - 9\epsilon} |\check{\alpha}|^2 \leq c_{13} r^{-1} \int_{\check{Y}} |\check{\alpha}|^2.$$

for some constant c_{13}.

If $k \geq 0$, it is easy to see that there exists a constant $c_{14} > 0$ such that

$$kg - mf \leq -c_{14}m \qquad \text{when } 9\epsilon \leq \rho \leq 2 - 9\epsilon.$$

With (5.8) and (5.5), there exists a constant c_{15} such that

$$\int_{9\epsilon \leq \rho \leq 2 - 9\epsilon} |\check{\alpha}|^2 \leq c_{15} r^{-1} \int_{\check{Y}} |\check{\alpha}|^2$$

provide $r \geq c_{15}$.

The above estimates implies that there exists a constant c_{16} such that

$$(5.16)\qquad \int_{\rho \geq 9\epsilon} |\check{\alpha}|^2 \leq c_{16} r^{-1} \int_{\check{Y}} |\check{\alpha}|^2$$

provided $r \geq c_{16}$.

Refined estimate on r. When $\rho \leq 10\epsilon$, the function $\frac{1}{\Delta}(mf' - kg')$ is identically equal to $\check{\gamma}_{k,m} = k + m$. By (5.1) and (5.16),

$$\int_{\rho \leq 9\epsilon} (r - \check{\gamma}_{k,m})^2 |\check{\alpha}|^2 \leq c_1 \int_{\check{Y}} |\check{\alpha}|^2 \leq c_1 (1 + \frac{c_{16} r^{-1}}{1 - c_{16} r^{-1}}) \int_{\rho \leq 9\epsilon} |\check{\alpha}|^2$$

Therefore, there exists a constant c_{17} such that

$$|r - \check{\gamma}_{k,m}| \leq c_{17} \tag{5.17}$$

provided $r \geq c_{17}$.

Zeroth order approximation of $\check{\alpha}$. When $\rho \leq 10\epsilon$, the operator $\mathcal{D}_2$ is

$$\mathcal{L}_{k,m} = 2\partial_z + \frac{\check{\gamma}_{k,m}}{2} \bar{z}$$

where $z = \rho e^{i\phi}$. If we regard $\mathcal{L}_{k,m}$ as an operator on $\mathbb{C}$, the theory of 2-dimensional harmonic oscillator applies. If $\check{\gamma}_{k,m} > 0$, $\mathcal{L}_{k,m}$ has the following properties: $\mathcal{C}^\infty(\mathbb{C})$ splits according to the frequency with respect to the S^1-action by $-i\partial_\phi = z\partial_z - \bar{z}\partial_{\bar{z}}$. For any $l \in \mathbb{Z}$, $\mathcal{L}_{k,m}$ maps the frequency l subspace to the frequency $l + 1$ subspace, and we only care about $\mathcal{L}_{k,m}$ on the frequency k subspace.

When $k < 0$, $\mathcal{L}_{k,m}$ has trivial kernel. It has a right inverse operator $G_{k,m}$ which maps the frequency $k+1$ subspace of $\mathcal{C}^\infty_{\text{cpt}}(\mathbb{C})$ to the frequency k subspace of $\mathcal{C}^\infty(\mathbb{C})$, and which satisfies

$$\int_{\mathbb{C}} |G_{k,m}\eta|^2 \leq \frac{1}{\check{\gamma}_{k,m}} \int_{\mathbb{C}} |\eta|^2$$

for any $\eta \in \mathcal{C}^\infty_{\text{cpt}}(\mathbb{C})$ with frequency $k + 1$.

When $k \geq 0$, $\mathcal{L}_{k,m}$ has a 1-dimensional kernel spanned by

$$\check{\xi}_{k,m} = (\frac{1}{k})^{\frac{1}{2}} (\frac{\check{\gamma}_{k,m}}{2})^{\frac{k+1}{2}} \frac{z^k}{\sqrt{\pi}} \exp(-\frac{\check{\gamma}_{k,m}}{4}|z|^2) \tag{5.18}$$

It has a right inverse operator $G_{k,m}$ satisfying

$$\int_{\mathbb{C}} |G_{k,m}\eta|^2 \leq \frac{1}{\check{\gamma}_{k,m}} \int_{\mathbb{C}} |\eta|^2 \qquad \text{, and} \qquad \int_{\mathbb{C}} \langle G_{k,m}\eta, \check{\xi}_{k,m} \rangle = 0$$

for any $\eta \in \mathcal{C}^\infty_{\text{cpt}}(\mathbb{C})$ with frequency $k + 1$.

Let χ_B be the cut-off function depending only on $\rho = |z|$, with $\chi_B(\rho) = 1$ when $\rho \leq 9\epsilon$ and $\chi_B(\rho) = 0$ when $\rho \geq 10\epsilon$. By (5.16) and (5.2), there exists a constant c_{18} such that

$$\int_{\check{Y}} |\mathcal{L}_{k,m}(\chi_B \check{\alpha})|^2 \leq c_{18} \int_{\check{Y}} |\check{\alpha}|^2 \tag{5.19}$$

provided $r \geq c_{18}$.

If $k < 0$, apply $G_{k,m}$ on (5.19) to find a constant c_{19} such that

$$\int_{\check{Y}} |\chi_B \check{\alpha}|^2 = \int_{\mathbb{C} \times S^1} |\chi_B \check{\alpha}|^2 \leq c_{19} r^{-1} \int_{\check{Y}} |\check{\alpha}|^2,$$

and this contradicts (5.16). Thus, k can only be nonnegative. If $k \geq 0$, apply $G_{k,m}$ on (5.19) to obtain a similar zeroth order approximation as in case 1. By the same token, we end with a contradiction to proposition 3.3.

Case 2 with $m < 0$. Similar estimates show that $\check{\alpha}$ peaks on $\rho \geq 2 - 9\epsilon$. When $\rho \geq 2 - 10\epsilon$, let $w = (2-\rho)e^{i\phi}$. The Dirac operator is

$$\frac{r-2}{2}\check{\alpha} + \frac{i}{2}(\partial_\phi\check{\alpha} - \partial_t\check{\alpha}) + \big(2\partial_w\check{\beta} + \frac{i}{2}\bar{w}(\partial_\phi\check{\beta} - \partial_t\check{\beta}) + \frac{\bar{w}}{2}\check{\beta}\big),$$
$$\big(-2\partial_{\bar{w}}\check{\alpha} + \frac{i}{2}w(\partial_\phi\check{\alpha} - \partial_t\check{\alpha})\big) - \frac{r+1}{2}\check{\beta} - \frac{i}{2}(\partial_\phi\check{\beta} - \partial_t\check{\beta}).$$

With the same argument, there can be at most one zero crossing happening near $\check{\gamma}_{k,m} = k - m$. End the proof of proposition 5.3. $\square$

5.2. **Second order approximation of eigensections.** We need a further understanding of eigensections in this model. We are going to obtain a second order approximation of eigensections with small eigenvalues. There are two ingredients. The first ingredient is that the true Dirac operator $\check{D}_r$ can have at most one small eigenvalue on $\mathcal{S}_{k,m}$. The second ingredient is an iteration scheme to build an approximation through the linearized operator. The following two lemmata serve the first ingredient.

Lemma 5.4. *There exists a constant $c > 1$ which has the following significance: Suppose that $\check{\psi}$ be a eigensection of $\check{D}_r$ for some $r \geq c$, and the magnitude of the corresponding eigenvalue is less than $\sqrt{\frac{r}{2}}$. Then*

$$\int_{\check{Y}} |\check{\beta}|^2 + r^{-1}\int_{\check{Y}} |\nabla_r\check{\beta}|^2 \leq cr^{-1}\int_{\check{Y}} |\check{\alpha}|^2.$$

Proof. The same as proof of proposition 3.1. $\square$

Lemma 5.5. *There exists a constant $c > 10$ which has the following significance: For any $r \geq c$, the Dirac operator $\check{D}_r$ on $\mathcal{S}_{k,m}$ has at most one eigenvalue whose magnitude is less than $\sqrt{\frac{r}{2}}$. If k is negative or $k = m = 0$, there is no such eigenvalue. Moreover, if there does exist such eigenvalue, then*

$$|r - \check{\gamma}_{k,m}| \leq c\sqrt{r}.$$

Proof. The proof is parallel to the proof of proposition 5.3. We just briefly mention where needs to be modified.

Suppose that $\psi \in \mathcal{S}_{k,m}$ is an eigensection whose eigenvalue has magnitude less than $\sqrt{\frac{r}{2}}$. By lemma 5.4, (5.1) and (5.2) would be replaced by

$$\int_{\check{Y}} \big|(r + \frac{kg' - mf'}{\Delta})\check{\alpha}\big|^2 \leq (r + c_1)\int_{\check{Y}} |\check{\alpha}|^2,$$
$$\int_{\check{Y}} \big|e^{i\phi}(\partial_\rho\check{\alpha} - \frac{kg - mf}{\Delta}\check{\alpha})\big|^2 \leq c_1\int_{\check{Y}} |\check{\alpha}|^2.$$

It is crucial that there is no $\mathcal{O}(r)$-term needed to bound the second component. Based on these two estimates, the bound in (5.3) becomes $c_2\sqrt{r}$. The estimates (5.6), (5.7), (5.8) and (5.9) remain the same. They imply that

- The refined estimate on r: $|r - \check{\gamma}_{k,m}| \leq c_3\sqrt{r}$ for some constant c_3.
- The same zeroth order approximation of $\check{\alpha}$.

If $\check{D}_r$ has two such eigensections, their orthogonality with lemma 5.4 implies that the inner product between their first components is small. It contradicts the zeroth order approximation of their first components. □

Suppose that $\check{D}_r$ on $\mathcal{S}_{k,m}$ has an eigenvalue λ_0 with $|\lambda_0| \leq 1$. Similar to (5.11) and (5.17), there exists a constant c_4 such that

$$|r - \check{\gamma}_{k,m}| \leq c_4 \tag{5.20}$$

provided $r \geq c_4$. Because of (5.20), there is no difference to state estimates in terms of r or $\check{\gamma}_{k,m}$. By lemma 5.5, if λ is an eigenvalue of $\check{D}_r$ on $\mathcal{S}_{k,m}$ other than λ_0, then

$$|\lambda - \lambda_0| \geq \sqrt{\frac{r}{8}}. \tag{5.21}$$

We are going to approximate the eigensection of λ_0 to second order. Again, we separate it into two cases according to whether

$$|m| < (\frac{1}{32\epsilon^2} - 1)k \qquad \text{or} \qquad |m| \geq (\frac{1}{32\epsilon^2} - 1)k.$$

Case 1. When $|m| < (\frac{1}{32\epsilon^2} - 1)k$, let

$$\begin{aligned} \check{\alpha} &= \check{\alpha}_{k,m}(\rho)e^{i(k\phi+mt)}\Delta^{-\frac{1}{2}}(2\pi)^{-1}, \\ \check{\beta} &= \check{\beta}_{k,m}(\rho)e^{i((k+1)\phi+mt)}\Delta^{-\frac{1}{2}}(2\pi)^{-1}, \end{aligned} \tag{5.22}$$

then the Dirac operator on $\check{\alpha}_{k,m}$ and $\check{\beta}_{k,m}$ is

(5.23)
$$\begin{aligned} &(\frac{r}{2} + \frac{kg' - mf'}{2\Delta})\check{\alpha}_{k,m} + (-\check{\beta}'_{k,m} - \frac{kg - mf}{\Delta}\check{\beta}_{k,m} - \frac{\Delta'}{2\Delta}\check{\beta}_{k,m}), \\ &(\check{\alpha}'_{k,m} - \frac{kg - mf}{\Delta}\check{\alpha}_{k,m} - \frac{\Delta'}{2\Delta}\check{\alpha}_{k,m}) - (\frac{r}{2} + 1 + \frac{kg' - mf'}{2\Delta} + \frac{f''g' - f'g''}{8\Delta})\check{\beta}_{k,m}. \end{aligned}$$

For simplicity, we change the variable by $x = \rho - \check{\rho}_{k,m}$. The Taylor series expansion of the eigensection equation at $x = 0$ is of the following form

$$\begin{aligned} \lambda\check{\alpha}_{k,m} &= (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2} + \mathfrak{r}_1 x^2 + \mathfrak{r}_3 x^3 + \mathfrak{R}_1^\alpha)\check{\alpha}_{k,m} \\ &\quad + (-\frac{\mathrm{d}}{\mathrm{d}x} + \check{\gamma}_{k,m}x + \mathfrak{c}_1 + \mathfrak{r}_2 x^2 + \mathfrak{R}_1^\beta)\check{\beta}_{k,m}, \\ \lambda\check{\beta}_{k,m} &= (\frac{\mathrm{d}}{\mathrm{d}x} + \check{\gamma}_{k,m}x + \mathfrak{c}_1 + \mathfrak{r}_2 x^2 + \mathfrak{c}_2 x + \mathfrak{r}_4 x^3 + \mathfrak{R}_2^\alpha)\check{\alpha}_{k,m} \\ &\quad - (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2} + \mathfrak{c}_3 + \mathfrak{r}_1 x^2 + \mathfrak{R}_2^\beta)\check{\beta}_{k,m}. \end{aligned} \tag{5.24}$$

By the Taylor's theorem, there exists a constant $c_5 > 0$ such that the coefficients $|\mathfrak{c}_j| \leq c_5$ and $|\mathfrak{r}_j| \leq c_5 r$; the remainder terms $|\mathfrak{R}^\alpha_j| \leq c_5(x^2 + rx^4)$ and $|\mathfrak{R}^\beta_j| \leq c_5(|x| + r|x|^3)$ for $|x| \leq \epsilon$. Note that these coefficients and the remainder terms only depends on k and m, but not r. The assumption on the existence of small eigenvalue implies that k and $|m|$ are less than some multiple of r.

Equation (5.24) can be used to find the higher order approximation of the eigensection by the following procedure: Rewrite the equation as

$$\begin{aligned}(-\frac{\mathrm{d}}{\mathrm{d}x} + \check{\gamma}_{k,m}x)\check{\beta}_{k,m} &= \mathfrak{F}_1(\check{\alpha}_{k,m}, \check{\beta}_{k,m}) + (\lambda - \frac{r}{2})\check{\alpha}_{k,m},\\ (\frac{\mathrm{d}}{\mathrm{d}x} + \check{\gamma}_{k,m}x)\check{\alpha}_{k,m} &= \mathfrak{F}_2(\check{\alpha}_{k,m}, \check{\beta}_{k,m}) + (\lambda + \frac{r}{2})\check{\beta}_{k,m}.\end{aligned}$$

Start with $\check{\alpha}_{k,m} = (\frac{\check{\gamma}_{k,m}}{\pi})^{\frac{1}{4}} \exp(-\frac{\check{\gamma}_{k,m}}{2}x^2)$, $\check{\beta}_{k,m} = 0$, and $\lambda = 0$. The next order term of $\check{\alpha}_{k,m}$ is solved by the second equation. The next order term of λ is determined by the condition that $\mathfrak{F}_1(\check{\alpha}_{k,m}, \check{\beta}_{k,m}) + (\lambda - \frac{r}{2})\check{\alpha}_{k,m}$ is orthogonal to $\exp(-\frac{\check{\gamma}_{k,m}}{2}x^2)$ with respect to the L^2-inner product. The next order term of $\check{\beta}_{k,m}$ is solved by the first equation.

Following this procedure, we have the second order approximation:

$$\begin{aligned}\check{\alpha}_{k,m} &= \big(1 + \mathfrak{a}_1(x) + \mathfrak{a}_2(x,r)\big)\chi(\epsilon x)\big(\frac{\check{\gamma}_{k,m}}{\pi}\big)^{\frac{1}{4}} \exp(-\frac{\check{\gamma}_{k,m}}{2}x^2),\\ \check{\beta}_{k,m} &= \big(\mathfrak{b}_1(x) + \mathfrak{b}_2(x)\big)\chi(\epsilon x)\big(\frac{\check{\gamma}_{k,m}}{\pi}\big)^{\frac{1}{4}} \exp(-\frac{\check{\gamma}_{k,m}}{2}x^2),\\ \lambda &= \frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2} + \frac{\mathfrak{r}_1}{2\check{\gamma}_{k,m}}\end{aligned}\tag{5.25}$$

where

$$\begin{aligned}\mathfrak{a}_1(x) &= -\mathfrak{c}_1 x - \frac{\mathfrak{r}_2}{3}x^3,\\ \mathfrak{a}_2(x,r) &= \Big(\frac{\mathfrak{c}_1^2 - \mathfrak{c}_2}{2} - \frac{\mathfrak{r}_1}{4\check{\gamma}_{k,m}}(r - \check{\gamma}_{k,m} + \frac{\mathfrak{r}_1}{2\check{\gamma}_{k,m}} + \mathfrak{c}_3)\Big)\big(x^2 - \frac{1}{2\check{\gamma}_{k,m}}\big)\\ &\quad + \big(\frac{\mathfrak{c}_1\mathfrak{r}_2}{3} - \frac{\mathfrak{r}_4}{4} - \frac{\mathfrak{r}_1^2}{8\check{\gamma}_{k,m}}\big)\big(x^4 - \frac{3}{4\check{\gamma}_{k,m}^2}\big) + \frac{\mathfrak{r}_2^2}{18}\big(x^6 - \frac{15}{8\check{\gamma}_{k,m}^3}\big),\\ \mathfrak{b}_1(x) &= -\frac{\mathfrak{r}_1}{2\check{\gamma}_{k,m}}x,\\ \mathfrak{b}_2(x) &= \big(\frac{\mathfrak{c}_1\mathfrak{r}_1 - \mathfrak{r}_3}{2\check{\gamma}_{k,m}} + \frac{\mathfrak{r}_1\mathfrak{r}_2}{4\check{\gamma}_{k,m}^2}\big)\big(x^2 + \frac{1}{2\check{\gamma}_{k,m}}\big) + \frac{\mathfrak{r}_1\mathfrak{r}_2}{6\check{\gamma}_{k,m}}x^4.\end{aligned}$$

The property needed for estimating the error term is that

$$\int_{\mathbb{R}} \big|x^l \exp(-\frac{\check{\gamma}_{k,m}}{2}x^2)\big|^2 \leq ((l+3)!)(\check{\gamma}_{k,m})^{-(l+\frac{1}{2})}$$

for any integer $l \geq 0$. If we plug (5.25) into (5.24), the size of the error term can be computed directly. Let $\check{\psi}_{n,m}$ be the section whose components are

given by (5.22) and (5.25), then there exists a constant c_6 such that

$$\int_{\check{Y}} \big|\check{D}_r\check{\psi}_{k,m} - (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2} + \frac{\mathfrak{r}_1}{2\check{\gamma}_{k,m}})\check{\psi}_{k,m}\big|^2 \le c_6 r^{-2}\int_{\check{Y}} |\check{\psi}_{k,m}|^2 \tag{5.26}$$

provided $r \ge c_6$. The properties of the second order approximation are summarized in the following proposition.

Proposition 5.6. *There exists a constant c which has the following significance: For any $r \ge c$ and $|m| < (\frac{1}{32\epsilon^2} - 1)k$, suppose that the Dirac operator $\check{D}_r$ on $\mathcal{S}_{k,m}$ has an eigenvalue λ_0 with $|\lambda_0| \le 1$. Then the corresponding eigensection is*

$$\check{\psi}^{\mathrm{eig}}_{k,m} = \check{\mathfrak{q}}_{k,m}\check{\psi}_{k,m} + \check{\psi}^{(3)}_{k,m}$$

with $\check{\psi}_{k,m}$ is given by (5.22) and (5.25), and

$$\int_{\check{Y}} |\check{\psi}^{\mathrm{eig}}_{k,m}|^2 = 1, \qquad \int_{\check{Y}} |\check{\psi}^{(3)}_{k,m}|^2 \le cr^{-3}, \qquad \textit{and} \qquad |\check{\mathfrak{q}}_{k,m} - 1| \le cr^{-1}.$$

Moreover,

$$\big|\lambda_0 - (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2} + \frac{\mathfrak{r}_1}{2\check{\gamma}_{k,m}})\big| \le cr^{-1}$$

where $\mathfrak{r}_1$ is the coefficient of the second order term in the Taylor series expansion of $\frac{1}{2\Delta}(kg' - mf')$ at $\check{\rho}_{k,m}$.

Proof. Let pr_{λ_0} be the L^2-orthogonal projection onto the eigenspace of λ_0, and write $\check{\psi}_{k,m}$ as $\mathrm{pr}_{\lambda_0}(\check{\psi}_{k,m}) + (\check{\psi}_{k,m} - \mathrm{pr}_{\lambda_0}(\check{\psi}_{k,m}))$. By (5.26) and (5.21),

$$\int_{\check{Y}} |\check{\psi}_{k,m} - \mathrm{pr}_{\lambda_0}(\check{\psi}_{k,m})|^2 \le 8c_6 r^{-3}\int_{\check{Y}} |\check{\psi}_{k,m}|^2.$$

From the expression (5.25), there exists a constant c_7 such that

$$\big|1 - \int_{\check{Y}} |\check{\psi}_{k,m}|^2\big| \le c_7 r^{-1}.$$

After normalizing the L^2-norm of $\check{\psi}_{k,m} + (\mathrm{pr}_{\lambda_0}(\check{\psi}_{k,m})) - \check{\psi}_{k,m})$, we obtain the desired expression of $\check{\psi}^{\mathrm{eig}}_{k,m}$.

The estimate on λ follows from (5.26). □

There is a subtlety about the section $\check{\psi}_{k,m}$ given by (5.25): It depends on r, in particular, the $\mathfrak{a}_2(x,r)$ term. However, that term is small, and it implies that the difference between eigensections at different r is small. More precisely, we have:

Corollary 5.7. *There exists a constant c which has the following significance: Suppose that $\check{D}_{r_1}$ and $\check{D}_{r_2}$ both have an eigenvalue whose magnitude is less than or equal to 1 on $\mathcal{S}_{k,m}$, and $r_1 \ge c$, $r_2 \ge c$ and $|m| < (\frac{1}{32\epsilon^2} - 1)k$.*

Let $\check\psi^{\text{eig}}_{k,m}(r_1)$ and $\check\psi^{\text{eig}}_{k,m}(r_2)$ be the corresponding eigensections of unit L^2-norm. Then

$$\Big|\int_{\check Y}\langle\check\eta,\check\psi^{\text{eig}}_{k,m}(r_1)\rangle\Big|\le c(\min\{r_1,r_2\})^{-\frac{3}{2}}\big(\int_{\check Y}|\check\eta|^2\big)^{\frac{1}{2}}$$

for any $\check\eta$ with $\int_\Sigma\langle\check\eta,\check\psi^{\text{eig}}_{k,m}(r_2)\rangle=0$.

Proof. By proposition 5.6, $|r_1-r_2|\le 8$. From the expression (5.25),

$$\begin{aligned}\int_{\check Y}\big|(\check\psi_{k,m}(r_1)-\check\psi_{k,m}(r_2))\big|^2&\le c_8(r_2-r_1)^2\int_{\mathbb R}x^4(\check\gamma_{k,m})^{\frac{1}{2}}\exp(-\check\gamma_{k,m}x^2)\\&\le c_9(\min\{r_1,r_2\})^{-4}\end{aligned}$$

for some constant c_8 and c_9. Since the $\mathfrak a_1(x)$-term and $\mathfrak b_1(x)$-term in (5.25) do not depend on r, there exists a constant c_{10} such that

$$\Big|1+(\frac{\mathfrak c_1^2}{2\check\gamma_{k,m}}+\frac{\mathfrak c_1\mathfrak r_2}{2\check\gamma_{k,m}^2}+\frac{5\mathfrak r_2^2}{24\check\gamma_{k,m}^3}+\frac{\mathfrak r_1^2}{8\check\gamma_{k,m}^3})-\int_{\check Y}|\check\psi_{k,m}(r_j)|^2\Big|\le c_{10}r_j^{-2}$$

for $j=1,2$. If we use this improved estimate in the proof of proposition 5.6, we conclude that

$$\big|\check{\mathfrak q}_{k,m}(r_1)-\check{\mathfrak q}_{k,m}(r_2)\big|\le c_{11}(\min\{r_1,r_2\})^{-2}$$

for some constant c_{11}. The difference between the eigensections is

$$\begin{aligned}\check\psi^{\text{eig}}_{k,m}(r_1)-\check\psi^{\text{eig}}_{k,m}(r_2)&=\big(\check{\mathfrak q}_{k,m}(r_1)-\check{\mathfrak q}_{k,m}(r_2)\big)\check\psi_{k,m}(r_1)+\check\psi^{(3)}_{k,m}(r_1)\\&\quad+\check{\mathfrak q}_{k,m}(r_1)\big(\check\psi_{k,m}(r_1)-\check\psi_{k,m}(r_2)\big)-\check\psi^{(3)}_{k,m}(r_2).\end{aligned}$$

Take the inner product of $\check\eta$ with the above expression, and integrate over $\check Y$. With the triangle inequality, the Cauchy–Schwarz inequality and proposition 5.6, we complete the proof of the corollary. □

Throughout the discussion for proposition 5.6, r is fixed. We can forget the assumption of proposition 5.6, and look at (5.25) independently. The inequality (5.26) still holds as long as r and $\gamma_{k,m}$ differ by $\mathcal O(1)$. In other words, it guarantees the existence of small eigenvalues.

Lemma 5.8. *There exist a constant c which has the following significance: For any k, m and r with $|m|<(\frac{1}{32\epsilon^2}-1)k$, $\gamma_{k,m}\ge c$ and $|r-\check\gamma_{k,m}+\frac{\mathfrak r_1}{\check\gamma_{k,m}}|\le 1$, the section $\check\psi_{k,m}$, defined by (5.22) and (5.25), satisfies*

$$\int_{\check Y}\big|\check D_r\check\psi_{k,m}-(\frac{r}{2}-\frac{\check\gamma_{k,m}}{2}+\frac{\mathfrak r_1}{2\check\gamma_{k,m}})\check\psi_{k,m}\big|^2\le cr^{-2}\int_{\check Y}|\check\psi_{k,m}|^2$$

with the same $\mathfrak r_1$ as that in proposition 5.6, and $\big|1-\int|\check\psi_{k,m}|^2\big|\le cr^{-1}$. Therefore, there exists an eigenvalue λ of $\check D_r$ on $\mathcal S_{k,m}$ with

$$\big|\lambda-(\frac{r}{2}-\frac{\check\gamma_{k,m}}{2}+\frac{\mathfrak r_1}{2\check\gamma_{k,m}})\big|\le cr^{-1}.$$

Proof. We only need to check that the coefficients and remainder terms of equation (5.24) remains the same order. It follows from the fact that k and $|m|$ are less than some multiple of $\gamma_{k,m}$: By the equation in definition 5.1, $k \leq c_{12}\gamma_{k,m}$ for some constant c_{12}, and $|m| < (\frac{1}{32\epsilon^2} - 1)c_{12}\gamma_{k,m}$. Hence, the bound on the coefficients and remainder terms of equation (5.24) remains the same order. □

Case 2. When $|m| \geq (\frac{1}{32\epsilon^2} - 1)k$, the linearized equation can be solved completely. Let us further assume that $m > 0$. The discussion for $m < 0$ is completely parallel, and will be omitted.

The functions $\check{\xi}_{k,m}$ defined by (5.18) are expected to be the almost eigensections. They are exponentially small when $\rho \geq 9\epsilon$.

Lemma 5.9. *There exist a constant c which has the following significance: For any integers k and m with $m \geq (\frac{1}{32\epsilon^2}-1)k \geq 0$, the function $\check{\xi}_{k,m}$ defined by (5.18) satisfies*

$$|\check{\xi}_{k,m}|^2 \leq c\exp(-\frac{\check{\gamma}_{k,m}}{c}|z|^2)$$

for any z with $|z| \geq 9\epsilon$.

Proof. Remember that $\check{\gamma}_{k,m}$ is $k + m$ when $m \geq (\frac{1}{32\epsilon^2} - 1)k \geq 0$. In this proof, we simply denote it by γ, and we are going to think γ as a variable with $\gamma \geq \frac{k}{32\epsilon^2}$. From the expression

$$4\pi^2|\check{\xi}_{k,m}|^2 = \frac{1}{k!}(\frac{\gamma}{2})^{k+1}|z|^{2k}\exp(-\frac{\gamma}{2}|z|^2),$$

we can see that for any fixed $\gamma \geq \frac{k}{32\epsilon^2}$ and $|z| \geq 9\epsilon$, it is monotone increasing in k. Thus,

$$\begin{aligned}
&4\pi^2|\check{\xi}_{k,m}|^2 \\
\leq&\frac{1}{\Gamma(32\epsilon^2\gamma+1)}(\frac{\gamma}{2})^{32\epsilon^2\gamma+1}|z|^{64\epsilon^2\gamma}\exp(-\frac{\gamma}{2}|z|^2) \\
=&\left(\frac{1}{\Gamma(32\epsilon^2\gamma+1)}(\frac{\gamma}{2})^{32\epsilon^2\gamma+1}|z|^{64\epsilon^2\gamma}\exp\big(-(\frac{1}{2}-\frac{1}{c})\gamma|z|^2\big)\right)\exp(-\frac{\gamma}{c}|z|^2)
\end{aligned}$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\mathrm{d}x$ is the standard gamma function. If c is sufficiently large, with the help of the Stirling's formula, it is not hard to see that the whole term in front of $\exp(-\frac{\gamma}{c}|z|^2)$ is uniformly bounded for all $\gamma > 0$ and $|z| \geq 9\epsilon$. □

Recall that χ_B is the cut-off function depending only on $\rho = |z|$, with $\chi_B(\rho) = 1$ when $\rho \leq 9\epsilon$ and $\chi_B(\rho) = 0$ when $\rho \geq 10\epsilon$. Let $\check{\psi}_{k,m}$ be the section whose first component is

$$\check{\alpha}_{k,m} = \chi_B\check{\xi}_{k,m}e^{imt}, \tag{5.27}$$

and whose second component is zero. By lemma 5.9 and (4.2), there exists a constant c_{13} such that

$$\int_{\check{Y}} \left|\check{D}_r \check{\psi}_{k,m} - (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2} - 1)\check{\psi}_{k,m}\right|^2 \leq c_{13} \exp(-\frac{r}{c_{13}}). \tag{5.28}$$

With (5.21) and (5.20), we conclude that:

Proposition 5.10. *There exists a constant c which has the following significance: For any $r \geq c$ and $m \geq (\frac{1}{32\epsilon^2} - 1)k \geq 0$, suppose that the Dirac operator $\check{D}_r$ on $\mathcal{S}_{k,m}$ has an eigenvalue λ_0 with $|\lambda_0| \leq 1$. Then the corresponding eigensection is*

$$\check{\psi}^{\mathrm{eig}}_{k,m} = \check{\mathfrak{q}}_{k,m} \check{\psi}_{k,m} + \check{\psi}^{(3)}_{k,m}$$

with $\check{\psi}_{k,m}$ given by (5.27), and

$$\int_{\check{Y}} |\check{\psi}^{\mathrm{eig}}_{k,m}|^2 = 1, \quad \int_{\check{Y}} |\check{\psi}^{(3)}_{k,m}|^2 \leq c \exp(-\frac{r}{c}), \quad \text{and} \quad |\check{\mathfrak{q}}_{k,m} - 1| \leq c \exp(-\frac{r}{c}).$$

Moreover,

$$\left|\lambda_0 - (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2})\right| \leq c \exp(-\frac{r}{c}).$$

In this case, the section $\check{\psi}_{k,m}$ does not depend on r, but $\check{\psi}^{\mathrm{eig}}_{k,m}$ does depend on r. However, the dependence is small. With the same argument for corollary 5.7, we have:

Corollary 5.11. *There exists a constant c which has the following significance: Suppose that $\check{D}_{r_1}$ and $\check{D}_{r_2}$ both have an eigenvalue whose magnitude is less than or equal to 1 on $\mathcal{S}_{k,m}$, and $r_1 \geq c$, $r_2 \geq c$ and $m \geq (\frac{1}{32\epsilon^2} - 1)k \geq 0$. Let $\check{\psi}^{\mathrm{eig}}_{k,m}(r_1)$ and $\check{\psi}^{\mathrm{eig}}_{k,m}(r_2)$ be the corresponding eigensections of unit L^2-norm. Then*

$$\left|\int_{\check{Y}} \langle \check{\eta}, \check{\psi}^{\mathrm{eig}}_{k,m}(r_1)\rangle\right| \leq c \exp(-\frac{\min\{r_1, r_2\}}{c}) \Big(\int_{\check{Y}} |\check{\eta}|^2\Big)^{\frac{1}{2}}$$

for any $\check{\eta}$ with $\int_\Sigma \langle \check{\eta}, \check{\psi}^{\mathrm{eig}}_{k,m}(r_2)\rangle = 0$.

In this case, the sections $\check{\psi}_{k,m}$ also guarantees the existence of small eigenvalues.

Lemma 5.12. *There exist a constant c which has the following significance: For any k, m and r with $m \geq (\frac{1}{32\epsilon^2} - 1)k \geq 0$, $\check{\gamma}_{k,m} \geq c$ and $|r - \gamma_{k,m}| \leq 1$, the section $\check{\psi}_{k,m}$ defined by (5.27) satisfies*

$$\int_{\check{Y}} \left|D_r \check{\psi}_{k,m} - (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2})\check{\psi}_{k,m}\right|^2 \leq c \exp(-\frac{r}{c}) \int_{\check{Y}} |\check{\psi}_{k,m}|^2$$

and $\left|1 - \int_{\check{Y}} |\check{\psi}_{k,m}|^2\right| \leq c \exp(-\frac{r}{c})$. Therefore, there exists an eigenvalue λ of $\check{D}_r$ on $\mathcal{S}_{k,m}$ with

$$\left|\lambda - (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2})\right| \leq c \exp(-\frac{r}{c}).$$

Since the coefficient $\mathfrak{r}_1$ in lemma 5.8 is equal to 0 for $m \geq (\frac{1}{32\epsilon^2} - 1)k \geq 0$, we can also formally put the $-\frac{\mathfrak{r}_1}{2\check{\gamma}_{k,m}}$ term in proposition 5.6 and lemma 5.12.

5.3. **Estimating the spectral flow function.** We now prove theorem 2.1 for this model case.

Theorem 5.13. *For the assocaited contact form (4.5) of the tubular neighborhood of the binding, there exists a constant c such that*

$$\left|\check{\mathrm{sf}}_a(r) - \frac{r^2}{4}\int_0^2 \Delta \mathrm{d}\rho\right| \leq cr$$

for all $r \geq c$. The function Δ is defined by (4.6).

Proof. Proposition 5.3 gives the following upper bound of the spectral flow function

$$\begin{aligned}\check{\mathrm{sf}}_a(r) &\leq \#\big\{\text{integers } k \geq 0 \text{ and } m,\ \text{with } \check{\gamma}_{k,m} \leq r + c_1\big\} + c_1 \\ &= \#\big\{\text{integers } k \geq 1 \text{ and } m,\ \text{with } \check{\gamma}_{k,m} \leq r + c_1\big\} + 2r + 3c_1\end{aligned}$$

for some constant c_1. In other words, we need to count the lattice points (k, m) on the right half-plane.

Consider the reparametrized polar coordinate (s, ρ) on the right half-plane: $k = s\big(f^2(\rho) + g^2(\rho)\big)^{-\frac{1}{2}} f(\rho)$, $m = s\big(f^2(\rho) + g^2(\rho)\big)^{-\frac{1}{2}} g(\rho)$. Let

$$\check{\gamma}(s, \rho) = 2s\big(f^2(\rho) + g^2(\rho)\big)^{-\frac{1}{2}},$$

then $\check{\gamma}(s, \rho)$ is equal to $\check{\gamma}_{k,m}$ for any lattice points (k, m). From the expression of $\check{\gamma}(s, \rho)$, it is not hard to see that there exists a constant $c_2 > 0$ such that the total number of lattice points with $\check{\gamma}_{k,m} \leq r$ is less than the area of where $\check{\gamma} \leq r + c_2$, and is greater than the area of where $\gamma \leq r - c_2$, provided $r \geq c_2$. The area can be evaluated directly

$$\int_0^2 \int_0^{\frac{\sqrt{f^2+g^2}}{2}(r+c_2)} \frac{\Delta}{f^2 + g^2} s \mathrm{d}s \mathrm{d}\rho = \frac{r^2 + 2c_2 r + c_2^2}{4} \int_0^2 \Delta \mathrm{d}\rho.$$

This proves the assertion on the upper bound of the spectral flow function.

Lemma 5.8 and lemma 5.12 finds a constant $c_3 > 0$ such that $\check{D}_r$ on $\mathcal{S}_{k,m}$ has a zero crossing happening within $[\check{\gamma}_{k,m} - c_3, \check{\gamma}_{k,m} + c_3]$, provided $\check{\gamma}_{k,m} \geq c_3$. Therefore,

$$\check{\mathrm{sf}}_a(r) \geq \#\{\text{integers } k \geq 0 \text{ and } m,\ \text{with } c_3 \leq \check{\gamma}_{k,m} \leq r + c_3\},$$

and the same area computation proves the lower bound on the spectral flow function. □

Theorem 5.13 can be used to find a sequence of numbers such that there are not too many zero crossings near these numbers. Here is the precise statement:

Lemma 5.14. *For any $\delta_3 > 0$, there exist a constant c and a sequence of numbers $\{s_n\}_{n\in\mathbb{N}}$ which depend on the associated contact forms, and which have the following significance:*

(i) *The total number of zero crossings of $\check{D}_r$ and $\tilde{D}_r$ happening between $s_n - \frac{\delta_3}{s_{n-1}}$ and $s_n + \frac{\delta_3}{s_{n-1}}$ is less than c for all $n \in \mathbb{N}$.*

(ii) *For each $n \in \mathbb{N}$, $|s_n - \gamma_n - \frac{1}{V}| \leq \frac{1}{4V}$. See (4.15) for γ_n.*

Proof. For $n \leq 8(\delta_3+1)V^2$, take s_n to be $\gamma_n + \frac{1}{V}$. For $n > 8(\delta_3+1)V^2$, we are going to define s_n inductively. By theorem 5.13, there exists a constant $c_1 > 0$ such that the total number of zero crossings happening within the interval

$$I_n = [\gamma_n + \frac{3}{4V}, \gamma_n + \frac{5}{4V}]$$

is less than $c_1 n \leq c_1 V s_{n-1}$. Divide I_n into sub-intervals whose lengths are $\frac{2\delta_3}{s_{n-1}}$. Then, the total number of sub-intervals is greater than $\frac{s_{n-1}}{8\delta_3 V}$. There must be a sub-interval which contains less than $8c_1\delta_3 V^2$ zero crossings. Let s_n be the midpoint of the sub-interval. From the construction, $\{s_n\}_{n\in\mathbb{N}}$ has the desired properties. □

5.4. **Higher order approximation on certain region.** This subsection is a remark on the approximation eigensections. On the region where $1-5\epsilon \leq \rho \leq 1+15\epsilon$ of $\check{Y}$, the function $f = V$ and $g = 2-\rho$, and the Dirac operator on $\mathcal{S}_{k,m}$ is already linear on this region, see (5.23).

If $\frac{k}{V}(1-11\epsilon) \leq m \leq \frac{k}{V}$, $\check{\rho}_{k,m}$ lies between 1 and $1+11\epsilon$, and $\check{\gamma}_{k,m}$ is equal to $\frac{2k}{V}$. All the higher order terms in (5.24) are equal to zero, and the correction terms in (5.25) are also equal to zero. Meanwhile, the almost solution (5.25) only supports within the interval $[1-\epsilon, 1+12\epsilon]$. Therefore, these almost solutions solve the Dirac equation up to an exponentially small error term. We have similar statements as proposition 5.10 and lemma 5.12.

Lemma 5.15. *There exists a constant c which has the following significance: For any $r \geq c$ and $\frac{k}{V}(1-11\epsilon) \leq m \leq \frac{k}{V}$, suppose that the Dirac operator $\check{D}_r$ on $\mathcal{S}_{k,m}$ has an eigenvalue λ_0 with $|\lambda_0| \leq 1$, then*

$$\left|\lambda_0 - (\frac{r}{2} - \frac{\check{\gamma}_{k,m}}{2})\right| \leq c\exp(-\frac{r}{c}).$$

On the other hand, if $\check{\gamma}_{k,m} \geq c$, $|r - \check{\gamma}_{k,m}| < 1$, and k and m satisfy the same constraint, the Dirac operator $\check{D}_r$ on $\mathcal{S}_{k,m}$ does have an eigenvalue λ_0 satisfying the above estimate.

Similarly, for the associated contact form of the Dehn-twist region, the Dirac operator is already linear on $[-1-15\epsilon, -1-5\epsilon]$ and $[1+5\epsilon, 1+15\epsilon]$.

Lemma 5.16. *There exists a constant c which has the following significance: For and $r \geq c$ and*

$$\frac{2(v-1-11\epsilon)}{V+2(v-1-11\epsilon)\tau(1)}k \leq m \leq \frac{2(v-1)}{V+2(v-1)\tau(1)}k \quad or$$
$$\frac{2(v+1)}{V+2(v+1)\tau(-1)}k \leq m \leq \frac{2(v+1+11\epsilon)}{V+2(v+1+11\epsilon)\tau(-1)}k,$$

suppose that the Dirac operator $\tilde{D}_r$ on $\mathcal{S}_{k,m}$ has an eigenvalue λ_0 with $|\lambda_0| \leq 1$, then

$$\left|\lambda_0 - (\frac{r}{2} - \frac{\tilde{\gamma}_{k,m}}{2})\right| \leq c\exp(-\frac{r}{c}).$$

On the other hand, if $\tilde{\gamma}_{k,m} \geq c$, $|r - \tilde{\gamma}_{k,m}| < 1$, and k and m satisfy the same constraint, the Dirac operator $\tilde{D}_r$ on $\mathcal{S}_{k,m}$ does have an eigenvalue λ_0 satisfying the above estimate.

For k and m in the first range, $\tilde{\gamma}_{k,m} = \frac{2(k+m\tau(1))}{V}$. For k and m in the second range, $\tilde{\gamma}_{k,m} = \frac{2(k+m\tau(-1))}{V}$.

5.5. **Contact forms with two S^1-symmetry.** The method in this section works for the contact forms with two global S^1-invariance. For instance, one can use the same method to prove theorem 2.1 for the overtwisted contact form in [T1], or some contact forms on T^3. There are two main differences:

(i) The frequency k might be negative.
(ii) There might be more than one zero crossing on each $\mathcal{S}_{k,m}$. But the number is decided by $f(\rho)$ and $g(\rho)$, and the zero modes peak at different region.

With this understood, the condition (2.5) is a shortcut for dealing the Dehn-twist region. It ensures the positivity of $\tilde{f}$. If $\tilde{f}$ is not always positive, we can still extend (the untwisting of) a to a contact form of the type (4.5) on $S^2 \times S^1$. The frequency k can be negative, and it requires some work to discuss it.

Here is a remark from the view point of contact topology: With the terminology of Giroux correspondence [G], our associated contact form is supported by an annulus with the identity map, and thus Stein fillable. If $\tilde{f}$ is not always positive, the extension ends up with an overtwisted contact form on $S^2 \times S^1$.

6. Lower bound of the spectral flow

We are going to prove a stronger statement which implies the lower bound in theorem 2.1. Before that,

Definition 6.1. Let us introduce the following notions:

(i) For the associated contact form of the tubular neighborhood of the binding, let $\check{I}(r', r)$ be the total number of zero crossings of $\check{D}_r$ happening within the interval $(r', r]$ and on $\mathcal{S}_{k,m}$ with $m \geq \frac{k}{V}$.

(ii) For the associated contact form of the Dehn-twist region, let $\tilde{I}(r', r)$ be the total number of zero crossings of $\tilde{D}_r$ happening within the interval $(r', r]$ and on $\mathcal{S}_{k,m}$ with

$$\frac{2(v-1)}{V-2(v-1)\tau(1)}k \leq m \leq \frac{2(v+1)}{V-2(v+1)\tau(-1)}k.$$

(iii) For any $n \in \mathbb{N}$, let $I_\Sigma(n)$ be the dimension of $\ker \bar{\partial}_n$.

If we fix $r' = 0$, these functions obey the following estimates:

Lemma 6.2. *There exists a constant $c > 0$ which has the following significance:*

$$\Big|\check{I}(0, r) - \frac{r^2}{4}\int_0^1 \Delta \mathrm{d}\rho\Big| \leq cr,$$

$$\Big|\tilde{I}(0, r) - \frac{r^2}{4}\int_{-1}^1 \tilde{\Delta} \mathrm{d}\rho\Big| \leq cr,$$

$$\Big|\sum_{n=1}^{[\frac{V}{2}r]} I_\Sigma(n) - \frac{Vr^2}{8\pi}\int_\Sigma \mathrm{d}\mu_\Sigma\Big| \leq cr$$

for all $r \geq c$. The functions Δ and $\tilde{\Delta}$ are defined by (4.6).

Proof. The proof for the first two inequalities is similar to the proof of theorem 5.13. We briefly explain it for $\check{I}(0, r)$, and adopt the same notation in the proof of theorem 5.13. There exists a constant c_1 such that $\check{I}(0, r)$ is less than

$$\begin{aligned}&\mathrm{Area}\big(\{\rho \leq 1 \text{ and } \check{\gamma}(s, \rho) \leq r + c_1\}\big) + c_1 r\\ &+ \mathrm{Area}\big(\{\frac{k}{V} \leq m \leq \frac{k}{V} + 1 \text{ and } \check{\gamma}(s, \rho) \leq r + c_1\}\big)\end{aligned}$$

for all $r \geq c_1$. The first term can be computed in the same way. It is easy to see that the last term is less than $c_2 r$ for some constant c_2.

The third inequality on $I_\Sigma(n)$ follows directly from the index formula (4.9) and lemma 4.2. □

Here comes the main theorem of this section:

Theorem 6.3. *There exist a constant c and a sequence of numbers $\{s_n\}_{n\in\mathbb{N}}$ which have the following significance:*

(i) *For all $n \geq 2$,*

$$\mathrm{sf}_a(s_n) - \mathrm{sf}_a(s_{n-1}) \geq I_\Sigma(n) + \check{I}(s_{n-1}, s_n) + \tilde{I}(s_{n-1}, s_n) - c.$$

(ii) *For each $n \in \mathbb{N}$, $|s_n - \gamma_n - \frac{1}{V}| \leq \frac{1}{4V}$, where $\gamma_n = \frac{2n}{V}$ as in (4.15).*

It is clear that the lower bound of the spectral flow function claimed in theorem 2.1 follows from theorem 6.3 and lemma 6.2.

We will prove theorem 6.3 by constructing almost eigensections. The following lemma measures the difference between true eigenvalues and almost

eigenvalues. It is an issue of linear algebra, but we state it for a Dirac operator.

Lemma 6.4. *Let $\mathcal{D}$ be a Dirac operator on the bundle $\mathbb{S}$. If there exist a constant δ_4, a finite number of smooth sections of $\mathbb{S}$, $\{\psi_l\}_{l=1}^L$, and real numbers $\{\mu_l\}_{l=1}^L$ with the properties:*

(i) *$\{\psi_l\}_{l=1}^L$ is an orthonormal set with respect to the L^2-inner product.*

(ii) *$\int |\mathcal{D}\psi_l - \mu_l\psi_l|^2 \leq \delta_4$ for all l.*

(iii) *$\int \langle \mathcal{D}\psi_l, \psi_{l'}\rangle = 0$ for all $l \neq l'$.*

(iv) $\sum_{\substack{1\leq l,l'\leq L \\ \text{and } l\neq l'}} |\int \langle \mathcal{D}\psi_l, \mathcal{D}\psi_{l'}\rangle| \leq \delta_4$.

Then, there exist L eigenvalues (counting multiplicity) of $\mathcal{D}$, $\{\lambda_l\}_{l=1}^L$, such that $|\lambda_l - \mu_l| \leq \sqrt{2\delta_4}$ for all l.

Proof. Clearly it is true for $L = 1$. Suppose the lemma holds for $L - 1$, we are going to show that it is true for L. Without loss of generality, assume that $\{\mu_l\}_{l=1}^L$ is nondecreasing in l.

For each $l \in \{1, 2, \cdots, L\}$, we can remove ψ_l and s_l, and apply the lemma. If there are L eigenvalues (counting multiplicity), we are done. If there are only $(L - 1)$ eigenvalues, $\{\lambda_l\}_{l=1}^{L-1}$, they must satisfy

$$|\lambda_l - \mu_l| \leq \sqrt{2\delta_4} \qquad \text{and} \qquad |\lambda_l - \mu_{l+1}| \leq \sqrt{2\delta_4}$$

for all $l \in \{1, 2, \cdots, L - 1\}$. The triangle inequalities implies that

$$|\mu_l - \mu_{l+1}| \leq 2\sqrt{2\delta_4}$$

for all $l \in \{1, 2, \cdots, L - 1\}$.

Suppose that $\{e_l\}_{l=1}^{L-1}$ are the eigensections corresponding to $\{\lambda_l\}_{l=1}^{L-1}$. There exist complex numbers $\{\mathfrak{c}_l\}_{l=1}^L$ such that $\sum_{l=1}^L |\mathfrak{c}_l|^2 = 1$, and $\sum_{l=1}^L \mathfrak{c}_l\psi_l$ is orthogonal to e_l for all $l \in \{1, 2, \cdots, L - 1\}$. For any real number ν, the operator $\mathcal{D} - \nu$ along $\sum_{l=1}^L \mathfrak{c}_l\psi_l$ satisfies the estimate:

$$\begin{aligned}
\int \big|(\mathcal{D} - \nu)(\sum_{l=1}^L \mathfrak{c}_l\psi_l)\big|^2 &\leq \sum_{l=1}^L |\mathfrak{c}_l|^2 \int \big|(\mathcal{D} - \nu)\psi_l\big|^2 + \sum_{\substack{1\leq l,l'\leq L \\ \text{and } l\neq l'}} \int \big|\langle \mathcal{D}\psi_l, \mathcal{D}\psi_{l'}\rangle\big| \\
&\leq \sum_{l=1}^L |\mathfrak{c}_l|^2(\sqrt{\delta_4} + |\nu - \mu_l|)^2 + \delta_4 \\
&\leq (\sqrt{2\delta_4} + \max_l |\nu - \mu_l|)^2.
\end{aligned}$$

Consider $\nu = \frac{\mu_1+\mu_L}{2}$, the above inequality finds another eigenvalue λ_L with

$$\mu_1 - \sqrt{2\delta_4} \leq \lambda_L \leq \mu_L + \sqrt{2\delta_4}.$$

Therefore, there exist some $l \in \{1, 2, \cdots, L\}$ such that $|\lambda_L - \mu_l| \leq \sqrt{2\delta_4}$. By re-numbering the indices of $\{\lambda_l\}_{l=1}^{L}$, these L eigenvalues satisfy the assertion of the lemma. □

The following proposition is the prototype of theorem 6.3.

Proposition 6.5. *There exist a constant $c > 0$ which has the following significance: For any integer $n \geq c$ and any $\delta_5^-, \delta_5^+ \in [\frac{1}{2V}, \frac{3}{2V}]$, let $\gamma_n = \frac{2n}{V}$ as in (4.15), then*

$$\begin{aligned}&\mathrm{sf}_a(\gamma_n + (\delta_5^+ + \frac{c}{\gamma_n})) - \mathrm{sf}_a(\gamma_n - (\delta_5^- + \frac{c}{\gamma_n})) \\ &\geq I_\Sigma(n) + \check{I}(\gamma_n - \delta_5^-, \gamma_n + \delta_5^+) + \tilde{I}(\gamma_n - \delta_5^-, \gamma_n + \delta_5^+)\end{aligned}$$

Proof. The proof contains three steps.

Step 1. This step constructs almost eigensections of D_{γ_n} from those three terms on the right hand side.

From the page: By proposition 4.4, for any $n \geq c_2$, there exists an orthonormal set of sections $\{\psi_{n,l}\}$ where $l \in \{1, 2, \cdots, I_\Sigma(n)\}$, with

$$\int_Y |D_{\gamma_n}\psi_{n,l}|^2 \leq c_2 \exp(-\frac{\gamma_n}{c_2}), \tag{6.1}$$

$\int_Y \langle D_{\gamma_n}\psi_{n,l}, \psi_{n,l'}\rangle = 0$ and $|\int_Y \langle D_{\gamma_n}\psi_{n,l}, D_{\gamma_n}\psi_{n,l'}\rangle| \leq c_2 \exp(-\frac{\gamma_n}{c_2})$ for all $l \neq l'$.

From the tubular neighborhood of the binding: If $\check{D}_r$ has a zero crossing at $\gamma \in (\gamma_n - \delta_5^-, \gamma_n + \delta_5^+]$ on $\mathcal{S}_{k,m}$ with $m \geq \frac{k}{V}$, proposition 5.6 and 5.10 for $r = \gamma$ find a constant c_3 such that

$$|\frac{\gamma}{2} - \frac{\check{\gamma}_{k,m}}{2} + \frac{\mathfrak{r}_1}{2\check{\gamma}_{k,m}}| \leq c_3\gamma^{-1} \leq 2c_3\gamma_n^{-1}$$

provided $n \geq c_3$. Then lemma 5.8 and lemma 5.12 for $r = \gamma_n$ find a constant c_4 and a section $\check{\psi}_{k,m}$ such that

$$\int_{\check{Y}} |\check{D}_{\gamma_n}\check{\psi}_{k,m} - (\frac{\gamma_n}{2} - \frac{\check{\gamma}_{k,m}}{2} + \frac{\mathfrak{r}_1}{2\check{\gamma}_{k,m}})\check{\psi}_{k,m}|^2 \leq c_4\gamma_n^{-2}\int_{\check{Y}} |\check{\psi}_{k,m}|^2$$

and $\left|1 - \int_{\check{Y}} |\check{\psi}_{k,m}|^2\right| \leq c_4\gamma_n^{-1}$. With the triangle inequality, we have

$$\int_Y |D_{\gamma_n}\check{\psi}_{k,m} - (\frac{\gamma_n}{2} - \frac{\gamma}{2})\check{\psi}_{k,m}|^2 \leq c_5\gamma_n^{-2}\int_Y |\check{\psi}_{k,m}|^2 \tag{6.2}$$

for some constant c_5. The section $\check{\psi}_{k,m}$ can be regarded as defined on Y.

From the Dehn-twist region: The same construction as the tubular neighborhood of the binding produces some sections $\tilde{\psi}_{k,m}$. After undoing the untwisting (4.3) on $\tilde{\psi}_{k,m}$, they can be regarded as defined on Y, and also satisfy (6.2).

Step 2. In order to apply lemma 6.4 on D_{γ_n} and the sections constructed in step 1, we need to check that they meet the requirements of lemma 6.4.

For requirement (i): The orthogonality is clear between any two sections constructed from the same region, and between one section from $\check{Y}$ and another section from $\tilde{Y}$.

For a section $\psi_{n,l}$ from the page and another section $\check{\psi}_{k,m}$ from $\check{Y}$, we only need to consider the case when $k = n$. For the section $\check{\psi}_{n,m}$, m is required to be greater than or equal to $\frac{n}{V}$, but it is complementary to the Atiyah–Patodi–Singer boundary condition for $\psi_{n,l}$, see (4.11) and (4.16). Therefore, $\int_Y \langle \psi_{n,l}, \check{\psi}_{k,m} \rangle = 0$.

For a section from the page and another section from $\tilde{Y}$, the argument is basically the same. At a first glance, the condition in definition 6.1 (ii) seems not to match with the boundary conditions (4.13) and (4.14). However, the untwisting operator (4.3) shifts the frequency, and a simple computation shows that these conditions are complementary to each other.

The lengths of some sections are not equal to 1, but almost. It can be easily fixed by a minor adjustment of the constants.

For requirement (ii): By (6.1), the almost eigenvalues of $\{\psi_{n,l}\}$ are all equal to 0. By (6.2), the almost eigenvalues are equal to $\frac{1}{2}(\gamma_n - \gamma)$ for $\check{\psi}_{k,m}$ and $\tilde{\psi}_{k,m}$, and γ is where the zero crossing happens of $\check{D}_r$ or $\tilde{D}_r$ on $\mathcal{S}_{k,m}$. The error term δ_4 is $c_6\gamma_n^{-2}$ for some constant c_6.

For requirement (iii): For any two sections from the page, it is given by proposition 4.4. The arguments for other situations are the same as that for requirement (i).

For requirement (iv): The pairing can only be nonzero between two sections from Σ. With the help of proposition 4.4, the summation is less than $2I_\Sigma(n)c_2\exp(-\frac{\gamma_n}{c_2}) \leq c_6\gamma_n^{-2}$.

Step 3. With these almost eigensections, lemma 6.4 finds the following eigenvalues for D_{γ_n}:

- There are $I_\Sigma(n)$ eigenvalues whose magnitude is less than $\sqrt{2c_6}\gamma_n^{-1}$.
- If $\check{I}(\gamma_n - \delta_5^-, \gamma_n + \delta_5^+)$ or $\tilde{I}(\gamma_n - \delta_5^-, \gamma_n + \delta_5^+)$ gets a count at γ, there is an eigenvalue λ associating to it, with
$$\left|\lambda - \frac{\gamma_n - \gamma}{2}\right| \leq \sqrt{2c_6}\gamma_n^{-1}.$$
- All the above eigenvalues are different.

Note that the magnitude of these eigenvalues are less than $\delta_5^+ + \delta_5^- \leq \frac{3}{C}$ for $n \geq 10c_5V^2$. Let c_7 be the constant given by corollary 3.2 for $\delta_1 = \frac{12}{V}$, then

$$\begin{aligned}&\mathrm{sf}_a(\gamma_n + (\delta_5^+ + \frac{\sqrt{2c_6} + c_7}{\gamma_n})) - \mathrm{sf}_a(\gamma_n - (\delta_5^- + \frac{\sqrt{2c_6} + c_7}{\gamma_n}))\\ \geq\ & I_\Sigma(n) + \check{I}(\gamma_n - \delta_5^-, \gamma_n + \delta_5^+) + \tilde{I}(\gamma_n - \delta_5^-, \gamma_n + \delta_5^+).\end{aligned}$$

This completes the proof of proposition 6.5. $\square$

We now prove the main theorem of this section.

Proof of thoerem 6.3. Let c_8 be the constant given by proposition 6.5. Lemma 5.14 with $\delta_3 = c_8$ finds a constant c_9 and a sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $|s_n - \gamma_n - \frac{1}{V}| \leq \frac{1}{4V}$ and

$$\begin{aligned} \check{I}(s_n - \frac{c_8}{s_{n-1}}, s_n + \frac{c_8}{s_{n-1}}) \leq c_9, \\ \tilde{I}(s_n - \frac{c_8}{s_{n-1}}, s_n + \frac{c_8}{s_{n-1}}) \leq c_9 \end{aligned} \tag{6.3}$$

for all $n \in \mathbb{N}$. If $n \geq 10c_8V^2$, $\delta_5^- = \gamma_n - s_{n-1} - \frac{c_8}{\gamma_n}$ and $\delta_5^+ = s_n - \gamma_n - \frac{c_8}{\gamma_n}$ meet the requirement of proposition 6.5. Hence,

$$\begin{aligned} &\mathrm{sf}_a(s_n) - \mathrm{sf}_a(s_{n-1}) \\ &\geq I_\Sigma(n) + \check{I}(s_{n-1} + \frac{c_8}{\gamma_n}, s_n - \frac{c_8}{\gamma_n}) + \tilde{I}(s_{n-1} + \frac{c_8}{\gamma_n}, s_n - \frac{c_8}{\gamma_n}) \\ &\geq I_\Sigma(n) + \check{I}(s_{n-1}, s_n) + \tilde{I}(s_{n-1}, s_n) - 4c_9 \end{aligned}$$

provided $n \geq 10c_8V^2$. The last inequality follows from (6.3) because $\gamma_n > s_{n-1} > s_{n-2}$. We complete the proof of theorem 6.3. □

7. The Dirac operator of trivial monodromy

In order to prove the upper bound of the spectral flow function, we need a further understanding of the Dirac operator on $\Sigma \times S^1$. In this section, we adopt the notations from subsection 4.4, and discuss the properties of the Cauchy–Riemann operator (4.8), which we denote by $\mathcal{D}_{r,n}$. Here are two estimates on $(\hat{\alpha}_n, \hat{\beta}_n) \in \mathcal{C}^\infty(\underline{\mathbb{C}} \oplus K_\Sigma^{-1})$ in terms of $\mathcal{D}_{r,n}(\hat{\alpha}_n, \hat{\beta}_n)$.

Lemma 7.1. *There exists a constant c which has the following significance: For any $r \geq c$ and integer n with $|r - \gamma_n| \geq r^{\frac{1}{2}}$,*

$$\int_\Sigma |\hat{\alpha}_n|^2 + |\hat{\beta}_n|^2 \leq cr^{-1} \int_\Sigma |\mathcal{D}_{r,n}(\hat{\alpha}_n, \hat{\beta}_n)|^2$$

for any $\hat{\alpha}_n$ and $\hat{\beta}_n$ satisfying the Atiyah–Patodi–Singer boundary condition for $\bar{\partial}_n$ and $\bar{\partial}_n^*$, respectively. γ_n is defined by (4.15).

Proof. Under the Atiyah–Patodi–Singer boundary condition, $\bar{\partial}_n$ and $\bar{\partial}_n^*$ are adjoint operators. Integration by parts formula gives

$$\begin{aligned} \int_\Sigma |\mathcal{D}_{r,n}(\hat{\alpha}_n, \hat{\beta}_n)|^2 = \int_\Sigma (\frac{r-\gamma_n}{2})^2|\hat{\alpha}_n|^2 + |\bar{\partial}_n^*\hat{\beta}_n|^2 - \langle \bar{\partial}_n^*\hat{\beta}_n, \hat{\alpha}_n\rangle \\ + |\bar{\partial}_n\hat{\alpha}_n|^2 + (\frac{r-\gamma_n-2}{2})^2|\hat{\beta}_n|^2 - \langle \bar{\partial}_n\hat{\alpha}_n, \hat{\beta}_n\rangle. \end{aligned} \tag{7.1}$$

With the Cauchy–Schwarz inequality, we complete the proof of this lemma. □

Proposition 7.2. *There exists a constant c which has the following significance: For any $r \geq c$ and integer n with $|r-\gamma_n| \leq r^{\frac{1}{2}}$, suppose that $\hat{\alpha}_n$ and $\hat{\beta}_n$ vanish near $\partial\Sigma$. Then*

$$\int_\Sigma \left|\hat{\alpha}_n - \hat{\mathrm{pr}}_n(\hat{\alpha}_n)\right|^2 + \left|\hat{\beta}_n\right|^2 \leq cr^{-1}\int_\Sigma \left|\mathcal{D}_{r,n}(\hat{\alpha}_n,\hat{\beta}_n)\right|^2$$

where $\hat{\mathrm{pr}}_n$ is the L^2-orthogonal projection onto the kernel of $\bar{\partial}_n$. Moreover, if $r \neq \gamma_n$,

$$\int_\Sigma \left|\hat{\mathrm{pr}}_n(\hat{\alpha}_n)\right|^2 \leq 4(r-\gamma_n)^{-2}\int_\Sigma \left|(\mathrm{pr}_1 \circ \mathcal{D}_{r,n})(\hat{\alpha}_n,\hat{\beta}_n)\right|^2$$

where pr_1 is the projection onto the first component.

Proof. The condition $|r-\gamma_n| \leq r^{\frac{1}{2}}$ implies that $n \geq \frac{V}{4}r$ provided $r \geq 4$. With lemma 4.2, there exists a constant $c_1 > 0$ such that

$$\int_\Sigma |\hat{\beta}_n|^2 \leq c_1 r^{-1}\int_\Sigma |\bar{\partial}_n^*\hat{\beta}_n|^2$$

for any $r \geq c_1$. Since the non-zero eigenvalues of $\bar{\partial}_n\bar{\partial}_n^*$ and $\bar{\partial}_n^*\bar{\partial}_n$ coincide,

$$\begin{aligned}\int_\Sigma |\hat{\alpha}_n - \hat{\mathrm{pr}}_n(\hat{\alpha}_n)|^2 &\leq c_1 r^{-1}\int_\Sigma \left|\bar{\partial}_n(\hat{\alpha}_n - \hat{\mathrm{pr}}_n(\hat{\alpha}_n))\right|^2 \\ &= c_1 r^{-1}\int_\Sigma |\bar{\partial}_n\hat{\alpha}_n|^2.\end{aligned} \tag{7.2}$$

See [APS, p.51 and p.56] for the discussion about boundary conditions and the property of the spectrum of $\bar{\partial}_n\bar{\partial}_n^*$ and $\bar{\partial}_n^*\bar{\partial}_n$. These two inequalities together with (7.1) give

$$\begin{aligned}\int_\Sigma |\mathcal{D}_{r,n}(\hat{\alpha}_n,\hat{\beta}_n)|^2 \geq \int_\Sigma &(\frac{r-\gamma_n}{2})^2|\hat{\alpha}_n|^2 + \frac{r}{c_1}|\hat{\beta}_n|^2 - 2|\hat{\beta}_n|^2 \\ &+ \frac{r}{2c_1}|\hat{\alpha}_n - \hat{\mathrm{pr}}_n(\hat{\alpha}_n)|^2 + (\frac{r-\gamma_n-2}{2})^2|\hat{\beta}_n|^2.\end{aligned}$$

This proves the first assertion of this lemma.

For the second assertion, note that

$$\begin{aligned}\mathcal{D}_{r,n}\big(\hat{\mathrm{pr}}_n(\hat{\alpha}_n),0\big) &= \Big(\frac{r-\gamma_n}{2}\hat{\mathrm{pr}}_n(\hat{\alpha}_n),0\Big), \\ \mathcal{D}_{r,n}\big(\hat{\alpha}_n - \hat{\mathrm{pr}}_n(\hat{\alpha}_n),\hat{\beta}_n\big) &= \Big(\frac{r-\gamma_n}{2}\big(\hat{\alpha}_n - \hat{\mathrm{pr}}_n(\hat{\alpha}_n)\big) + \bar{\partial}_n^*\check{\beta}_n,\cdots\Big).\end{aligned}$$

Therefore, $\mathcal{D}_{r,n}$ preserves the L^2-orthogonality between $\big(\hat{\mathrm{pr}}_n(\hat{\alpha}_n),0\big)$ and $\big(\hat{\alpha}_n - \hat{\mathrm{pr}}_n(\hat{\alpha}_n),\hat{\beta}_n\big)$. This leads to the desired estimate on $\hat{\mathrm{pr}}_n(\hat{\alpha}_n)$. ∎

In next section, we are going to study the zero modes on Y through $\Sigma\times S^1$, $\check{Y}$ and $\tilde{Y}$. The cut-off function would cause some overlaps of these models. To tackle this issue, we need to study $\ker\bar{\partial}_n$ carefully.

As discussed in subsection 4.4, especially the part after definition 4.3, any solution of $\bar{\partial}_n$ on $\Sigma_{-11\epsilon}$ naturally extends to a solution on Σ, and still

satisfies the corresponding Atiyah–Patodi–Singer boundary condition. Let $\ker_0 \bar{\partial}_n$ be the subspace of $\ker \bar{\partial}_n$ which are extended from $\Sigma_{-11\epsilon}$. Consider the following sections which peak on $\Sigma \backslash \Sigma_{-11\epsilon}$.

Adjacent to the tubular neighborhood of the binding. For any integers $n > 0$ and m with $\frac{n}{V}(1-11\epsilon) \leq m < \frac{n}{V}$, let

$$\check{\zeta}_{n,m} = \chi\big(\epsilon(\rho - 2 + \frac{Vm}{n})\big)\big(\frac{2n}{V\pi^3}\big)^{\frac{1}{4}} \exp\big(-\frac{n}{V}(\rho - 2 + \frac{Vm}{n})^2\big)e^{imt}. \tag{7.3}$$

Adjacent to the Dehn-twist region. For any integers $n > 0$ and m with $\frac{2n}{V}(v+1+11\epsilon) \geq m > \frac{2n}{V}(v+1)$ or $\frac{2n}{V}(v-1-11\epsilon) \leq m < \frac{2n}{V}(v-1)$, let

$$\tilde{\zeta}_{n,m} = \chi\big(\epsilon(\rho - v + \frac{Vm}{2n})\big)\big(\frac{n}{2V\pi^3}\big)^{\frac{1}{4}} \exp\big(-\frac{2n}{V}(\rho - v + \frac{Vm}{2n})^2\big)e^{imt}. \tag{7.4}$$

Lemma 7.3. *There exists a constant $c > 0$ which has the following significance: For any $n \geq c$, the kernel of $\bar{\partial}_n$ has the orthonormal basis*

$$\{\textit{orthornormal basis of } \ker_0 \bar{\partial}_n\} \oplus \{\check{\mathfrak{p}}_{n,m}\check{\zeta}_{n,m} + \check{\zeta}^{\mathrm{rem}}_{n,m}\} \oplus \{\tilde{\mathfrak{p}}_{n,m}\tilde{\zeta}_{n,m} + \tilde{\zeta}^{\mathrm{rem}}_{n,m}\}$$

with respect to the L^2-inner product on Σ. The range of m for the second summand is $\{\frac{n}{V}(1-11\epsilon) \leq m < \frac{n}{V}\}$; the range of m for the third summand is $\{\frac{2n}{V}(v+1+11\epsilon) \geq m > \frac{2n}{V}(v+1)\}$ and $\{\frac{2n}{V}(v-1-11\epsilon) \leq m < \frac{2n}{V}(v-1)\}$. The elements in the decomposition have the following features:

(i) $\check{\mathfrak{p}}_{n,m}$ *is a constant between $\frac{1}{2}$ and 2, and $\int_\Sigma |\check{\zeta}^{\mathrm{rem}}_{n,m}|^2 \leq c\exp(-\frac{n}{c})$.*

(ii) $\tilde{\mathfrak{p}}_{n,m}$ *is a constant between $\frac{1}{2}$ and 2, and $\int_\Sigma |\tilde{\zeta}^{\mathrm{rem}}_{n,m}|^2 \leq c\exp(-\frac{n}{c})$.*

(iii) *For any $\alpha_n \in \ker_0 \bar{\partial}_n$, $\int_{\Sigma\backslash\Sigma_{-2\epsilon}} |\alpha_n|^2 \leq c\exp(-\frac{n}{c}) \int_\Sigma |\alpha_n|^2$.*

Proof. Consider the orthogonal set:

$$\{\text{orthonormal basis of } \ker_0 \bar{\partial}_n\} \oplus \{\check{\zeta}_{n,m}\} \oplus \{\tilde{\zeta}_{n,m}\}.$$

By dimension counting, the total number of elements in this set is equal to the dimension of $\ker \bar{\partial}_n$. It is easy to see that $\frac{1}{3} \leq \int_\Sigma |\check{\zeta}_{n,m}|^2 \leq 1$ and $\frac{1}{3} \leq \int_\Sigma |\tilde{\zeta}_{n,m}|^2 \leq 1$. The elements in the last two summand are not annihilated by $\bar{\partial}_n$, and we are going to modify them by the following procedure:

Start with an orthonormal basis of $\ker_0 \bar{\partial}_n$, and pick any $\check{\zeta}_{n,m}$. A direct computation shows that there exists a constant c_2 such that

$$\int_\Sigma |\bar{\partial}_n \check{\zeta}_{n,m}|^2 \leq c_2 \exp(-\frac{n}{c_2}).$$

Let $\check{\mathrm{pr}}_n(\check{\zeta}_{n,m})$ be the L^2-orthogonal projection of $\check{\zeta}_{n,m}$ onto $\ker \bar{\partial}_n$. With (7.2), there exists a constant c_3 such that

$$\int_\Sigma \big|(\check{\zeta}_{n,m} - \check{\mathrm{pr}}_n(\check{\zeta}_{n,m}))\big|^2 \leq c_3 \exp(-\frac{n}{c_3}).$$

If we apply the Gram–Schmidt process on

$$\check{\mathrm{pr}}_n(\check{\zeta}_{n,m}) = \check{\zeta}_{n,m} + (\check{\mathrm{pr}}_n(\check{\zeta}_{n,m}) - \check{\zeta}_{n,m})$$

with the orthonormal basis of $\ker_0 \bar{\partial}_n$, the output would be $\check{\mathfrak{p}}_{n,m}\check{\zeta}_{n,m} + \check{\zeta}^{\mathrm{rem}}_{n,m}$ with property (i).

We can keep doing this projection and Gram–Schmidt process. Since the total number of steps is less than n, the error term is always less than $c_4 \exp(-\frac{n}{c_4})$ for some constant c_4. It gives the orthonormal basis for $\ker \bar{\partial}_n$ with property (i) and (ii).

The proof of property (iii) is the same as that for proposition 4.4 $\square$

7.1. **Without invoking the Atiyah–Patodi–Singer theory.** This subsection is a remark, and is not directly related to other parts of this paper.

We provide another approach to the Dirac operator (4.8) on $\Sigma \times S^1$. The idea is to compactify $\Sigma \times S^1$ such that the classical Riemann-Roch theorem applies. If we divide the contact form by V, it becomes

$$\mathrm{d}\phi + \frac{2}{V}\mu_\Sigma. \tag{7.5}$$

For each boundary component, consider the following handle attaching by $D^2 \times S^1$, where D^2 is the unit disk.

7.1.1. *Adjacent to the tubular neighborhood of the binding.* In terms of the coordinate in subsection 4.1, Σ is defined by $\rho \geq 1$. When $1 \leq \rho \leq 1 + 15\epsilon$, the 1-form $2\mu_\Sigma$ is $(2-\rho)\mathrm{d}t$. Choose an integer $N_o > 1 + \frac{1}{V}$, and a smooth function $h_o(\rho)$ defined on $[0, 1]$ with

$$h_o(\rho) = \begin{cases} -\rho^2 & \text{when } \rho \in [0, 10\epsilon] \\ \frac{2-\rho}{V} - N_o & \text{when } \rho \in [1-5\epsilon, 1] \end{cases}$$

and $h_o'(\rho) < 0$ except at $\rho = 0$. The attaching handle $D^2 \times S^1$ has coordinate $(\rho e^{it}, e^{i\phi_o})$. When $\rho = 1$, the identification is

$$\begin{array}{ccc} \partial D^2 \times S^1 & \longleftrightarrow & \partial\Sigma \times S^1 \\ (e^{it}, e^{i\phi_o}) & \sim & (e^{it}, e^{i(\phi_o - N_o t)}) \end{array}$$

The 1-form (7.5) has the following extension to the handle

$$\mathrm{d}\phi_o + h_o(\rho)\mathrm{d}t. \tag{7.6}$$

7.1.2. *Adjacent to the Dehn-twist region: $\rho \geq 1$.* In terms of the coordinate in subsection 4.2, Σ is defined by $\rho \geq 1$ or $\rho \leq -1$. We first discuss the handle attaching to the part where $\rho \geq 1$. When $1 \leq \rho \leq 1 + 15\epsilon$, the 1-form $2\mu_\Sigma$ is $2(v-\rho)\mathrm{d}t$. Choose an integer $N_+ > 1 + \frac{2(v-1)}{V}$, and a smooth function $h_+(\rho)$ defined on $[0, 1]$ with

$$h_+(\rho) = \begin{cases} -\rho^2 & \text{when } \rho \in [0, 10\epsilon] \\ \frac{2(v-\rho)}{V} - N_+ & \text{when } \rho \in [1-5\epsilon, 1] \end{cases}$$

and $h_+'(\rho) < 0$ except at $\rho = 0$. The attaching handle $D^2 \times S^1$ has coordinate $(\rho e^{it}, e^{i\phi_+})$. When $\rho = 1$, the identification is

$$\begin{array}{ccc} \partial D^2 \times S^1 & \longleftrightarrow & \partial\Sigma \times S^1 \\ (e^{it}, e^{i\phi_+}) & \sim & (e^{it}, e^{i(\phi_+ - N_+ t)}) \end{array}$$

The 1-form (7.5) has the following extension to the handle

$$\mathrm{d}\phi_+ + h_+(\rho)\mathrm{d}t. \tag{7.7}$$

7.1.3. *Adjacent to the Dehn-twist region: $\rho \le -1$.* When $-1-15\epsilon \le \rho \le -1$, the 1-form $2\mu_\Sigma$ is $2(v-\rho)\mathrm{d}t$. Choose an integer $N_- > 1 - \frac{2(v+1)}{V}$, and a smooth function $h_-(\rho)$ defined on $[-1, 0]$ with

$$h_-(\rho) = \begin{cases} \rho^2 & \text{when } \rho \in [-10\epsilon, 0] \\ \frac{2(v-\rho)}{V} + N_- & \text{when } \rho \in [-1+5\epsilon, -1] \end{cases}$$

and $h_-'(\rho) > 0$ except at $\rho = 0$. The attaching handle $D^2 \times S^1$ has coordinate $(\rho e^{it}, e^{i\phi_-})$ with $\rho \in [-1, 0]$. When $\rho = -1$, the identification is

$$\begin{array}{ccc} \partial D^2 \times S^1 & \longleftrightarrow & \partial\Sigma \times S^1 \\ (e^{it}, e^{i\phi_-}) & \sim & (e^{it}, e^{i(\phi_- + N_- t)}) \end{array}$$

The 1-form (7.5) has the following extension to the handle

$$\mathrm{d}\phi_- + h_-(\rho)\mathrm{d}t. \tag{7.8}$$

Denote the 3-manifold by $\mathfrak{L}$. If we forget the S^1-component and attach those D^2 to Σ, we obtain a closed surface Σ^{c}. The 3-manifold $\mathfrak{L}$ is a $U(1)$-bundle over Σ^{c}. The degree of $\mathfrak{L}$ is $N_o + N_+ + N_-$, and it carries a connection given by (7.5), (7.6), (7.7) and (7.8).

The operators $\bar\partial_n$ and $\bar\partial_n^*$ is the classical Cauchy–Riemann operator on $\mathfrak{L}^n \oplus (\mathfrak{L}^n \otimes K_{\Sigma^{\mathrm{c}}}^{-1})$, and the Riemann–Roch theorem applies. Moreover, when $n \to \infty$, the behavior of holomorphic sections on the attaching handles can be studied in the same way as section 5. It allows us to remove solutions which peak on the attaching handles.

8. Upper bound of the spectral flow

Here comes the main theorem of this section. With lemma 6.2, it implies the upper bound of the spectral flow function claimed in theorem 2.1.

Theorem 8.1. *There exist a constant c and a sequence of numbers $\{s_n\}_{n\in\mathbb{N}}$ which have the following significance:*

(i) *For all $n \ge 2$,*

$$\mathrm{sf}_a(s_n) - \mathrm{sf}_a(s_{n-1}) \le I_\Sigma(n) + \check{I}(s_{n-1}, s_n) + \tilde{I}(s_{n-1}, s_n) + c.$$

(ii) *For each $n \in \mathbb{N}$, $|s_n - \gamma_n - \frac{1}{V}| \le \frac{1}{4V}$, where γ_n is defined by (4.15).*

The basic strategy for proving the theorem is to project true zero modes onto the vector space spanned by certain almost eigensections. The following lemma is the technical tool to bound the total number of zero modes. It is implied by the Welch bound [W]. We also include its proof for completeness.

Lemma 8.2. *For any $\delta_6 > 0$, there exists a constant c which has the following significance: For any integer $L_o > 2\delta_6$, suppose that $\{u_l\}_{l=1}^{L}$ is a set of unit vectors in $\mathbb{C}^{L_o}$ such that their inner product is small in the following sense*

$$\big|\langle u_l, u_{l'}\rangle\big| < \frac{\delta_6}{L_o} \qquad \text{for all } l \neq l'.$$

Then $L \leq L_o + c$.

Proof. We only need to consider the case when $L > L_o$. Let U be the $L \times L_o$ matrix whose l-th column is the vector u_l. Consider the matrix $H = U^*U$. The matrix H is Hermitian, and the dimension of its kernel is at least $L - L_o$. Suppose that the eigenvalues of H are $\{\lambda_1, \cdots, \lambda_{L_o}, 0, \cdots, 0\}$, then $\lambda_1 + \cdots + \lambda_{L_o} = L$. With the Cauchy–Schwarz inequality,

$$\begin{aligned} L^2 &= (\lambda_1 + \cdots + \lambda_{L_o})^2 \\ &\leq L_o(\lambda_1^2 + \cdots + \lambda_{L_o}^2) = L_o \operatorname{trace}(H^*H) \\ &= L_o\big(L + \sum_{l\neq l'} \big|\langle u_l, u_{l'}\rangle\big|^2\big) \\ &\leq L_o(L + L(L-1)\frac{\delta_6^2}{L_o^2}). \end{aligned}$$

It implies that $L \leq L_o + c$. ∎

The remainder of this section is devoted to the proof of theorem 8.1.

Proof of theorem 8.1. This proof contains ten steps.

Step 1. This step constructs the sequence $\{s_n\}_{n\in\mathbb{N}}$. Let c_1 be the constant given by corollary 3.2 for the associated contact forms with $\delta_1 = 1$. In particular, for any $r \geq c_1$, if $\check{D}_r$ or $\tilde{D}_r$ has an eigenvalue λ_0 with $|\lambda_0| \leq r^{-1}$ on $\mathcal{S}_{k,m}$, there is a zero crossing on $\mathcal{S}_{k,m}$ happening somewhere in the interval

$$[r - \frac{c_1+2}{r}, r + \frac{c_1+2}{r}].$$

Proposition 5.3 says that it is the only zero crossing on $\mathcal{S}_{k,m}$ for $r \geq c_1$.

We then invoke lemma 5.14 for $\delta_3 = c_1 + 2$ to obtain a sequence $\{s_n\}_{n\in\mathbb{N}}$ and a constant $c_2 > 0$ such that

(i) The total number of zero crossings of $\check{D}_r$ and $\tilde{D}_r$ happening between $s_n - \frac{c_1+2}{s_{n-1}}$ and $s_n + \frac{c_1+2}{s_{n-1}}$ is less than c_2 for any $n \in \mathbb{N}$.

(ii) For each $n \in \mathbb{N}$, $|s_n - \gamma_n - \frac{1}{V}| \leq \frac{1}{4V}$. See (4.15) for γ_n.

Step 2. This step introduces the index sets of the vector space of almost eigensections. For each $n \in \mathbb{N}$, consider the following condition on k and m

$$\min_{r\in[s_{n-1},s_n]} \min_{\substack{\lambda\in\mathrm{Spec}\check{D}_r \\ \text{on } \mathcal{S}_{k,m}}} |\lambda| \le \frac{1}{s_n}, \tag{8.1}$$

and the index sets:

(i) $\check{E}^o_n$ is the set of the $(k,m)\in\mathbb{Z}^2$ satisfying $m \ge \frac{k}{C}$ and condition (8.1);

(ii) $\check{E}_n$ is the set of the $(k,m)\in\mathbb{Z}^2$ satisfying $m \ge \frac{k}{C}(1-11\epsilon)$ and condition (8.1);

(iii) $\tilde{E}^o_n$ is the set of the $(k,m)\in\mathbb{Z}^2$ satisfying

$$\frac{2(v-1)}{V-2(v-1)\tau(1)}k \le m \le \frac{2(v+1)}{V-2(v+1)\tau(-1)}k$$

and condition (8.1) for $\tilde{D}_r$;

(iv) $\tilde{E}_n$ is the set of the $(k,m)\in\mathbb{Z}^2$ satisfying

$$\frac{2(v-1-11\epsilon)}{V-2(v-1-11\epsilon)\tau(1)}k \le m \le \frac{2(v+1+11\epsilon)}{V-2(v+1+11\epsilon)\tau(-1)}k$$

and condition (8.1) for $\tilde{D}_r$.

From the construction in step 1, there exists a constant c_3 such that

$$\begin{aligned} &\check{I}(s_{n-1},s_n) \le \#\check{E}^o_n \le \check{I}(s_{n-1},s_n)+2c_2 \quad \text{, and} \quad \check{I}(s_{n-1},s_n) \le c_3 s_n, \\ &\tilde{I}(s_{n-1},s_n) \le \#\tilde{E}^o_n \le \tilde{I}(s_{n-1},s_n)+2c_2 \quad \text{, and} \quad \tilde{I}(s_{n-1},s_n) \le c_3 s_n \end{aligned} \tag{8.2}$$

for all $n \ge c_3$. See definition 6.1 for $\check{I}$ and $\tilde{I}$.

With these estimates, theorem 8.1 is equivalent to the following *claim*: There exists a constant c_4 such that

$$\mathrm{sf}_a(s_n) - \mathrm{sf}_a(s_{n-1}) \le I_\Sigma(n) + \#\check{E}^o_n + \#\tilde{E}^o_n + c_4 \tag{8.3}$$

for any $n \ge c_4$. The proof of this claim occupies step 3 to step 10.

Lemma 5.15 and lemma 5.16 characterize $\check{E}_n\backslash\check{E}^o_n$ and $\tilde{E}_n\backslash\tilde{E}^o_n$ completely. There exists a constant $c_5>0$ such that

(8.4)

$$\begin{aligned} \check{E}_n\backslash\check{E}^o_n &= \big\{(k,m)\big| k=n, \text{ and } \frac{n}{C}(1-11\epsilon) \le m < \frac{n}{C}\big\} \quad \text{, and} \\ \tilde{E}_n\backslash\tilde{E}^o_n &= \big\{(k,m)\big| k+mh(1)=n, \frac{2n}{C}(v-1-11\epsilon) \le m < \frac{2n}{C}(v-1)\big\} \\ &\quad \cup \big\{(k,m)\big| k+mh(-1)=n, \frac{2n}{C}(v+1) < m \le \frac{2n}{C}(v+1+11\epsilon)\big\} \end{aligned}$$

for any $n \ge c_5$.

Step 3. This step defines six cut-off functions. Let $\check{\chi}^o$ and $\check{\chi}$ be the cut-off functions whose supports belong to the tubular neighborhood of the binding, and which depend only on ρ in terms of the coordinate in subsection 4.1, with

$$\check{\chi}^o(\rho) = \begin{cases} 1 & \text{when } \rho \leq 1 + 4\epsilon \\ 0 & \text{when } \rho \geq 1 + 6\epsilon \end{cases}, \qquad \check{\chi}(\rho) = \begin{cases} 1 & \text{when } \rho \leq 1 + 8\epsilon \\ 0 & \text{when } \rho \geq 1 + 10\epsilon \end{cases}.$$

Note that $\check{\chi} \circ \check{\chi}^o = \check{\chi}^o$.

Let $\tilde{\chi}^o$ and $\tilde{\chi}$ be the cut-off functions whose supports belong to the Dehn-twist region, and which depend only on ρ in terms of the coordinate in subsection 4.2, with

$$\tilde{\chi}^o(\rho) = \begin{cases} 1 & \text{when } |\rho| \leq 1 + 4\epsilon \\ 0 & \text{when } |\rho| \geq 1 + 6\epsilon \end{cases}, \qquad \tilde{\chi}(\rho) = \begin{cases} 1 & \text{when } |\rho| \leq 1 + 8\epsilon \\ 0 & \text{when } |\rho| \geq 1 + 10\epsilon \end{cases}.$$

Note that $\tilde{\chi} \circ \tilde{\chi}^o = \tilde{\chi}^o$.

Let $\hat{\chi}^o$ be $1 - \check{\chi}^o - \tilde{\chi}^o$. In terms of the terminology introduced by definition 4.3, $\hat{\chi}^o = 1$ on $\Sigma_{-6\epsilon} \times S^1$, and $\hat{\chi}^o = 0$ on $Y \backslash (\Sigma_{-4\epsilon} \times S^1)$. Let $\hat{\chi}$ be the similar cut-off function depending only on ρ near $\partial\Sigma \times S^1$, with

$$\hat{\chi} = \begin{cases} 1 & \text{on } \Sigma_{-2\epsilon} \times S^1 \\ 0 & \text{on } Y \backslash (\Sigma \times S^1) \end{cases}.$$

Also, $\hat{\chi} \circ \hat{\chi}^o = \hat{\chi}^o$.

Step 4. For any zero mode ψ of D_r, we are going to study $\check{\chi}\psi$ and $\tilde{\chi}\psi$ in terms of the eigensections on $\check{Y}$ and $\tilde{Y}$.

With the results in subsection 5.2, there exists a constant c_6 which has the following significance: Suppose that $n \geq c_6$, $r \in (s_{n-1}, s_n]$, and $(k, m) \in \check{E}_n$, then $\check{D}_r$ has a unique eigenvalue λ_0 on $\mathcal{S}_{k,m}$ with $|\lambda_0| < 1$. With this understood, for any $r \geq c_6 + 1$, let $\check{\mathrm{pr}}_r$ be the L^2-orthogonal projection onto the eigenspaces of small eigenvalues arising from $\check{E}_n$. Similarly, let $\tilde{\mathrm{pr}}_r$ be the L^2-orthogonal projection onto the eigenspaces of small eigenvalues arising from $\tilde{E}_n$.

Lemma 8.3. *There exists a constant c which has the following significance: For any $n \geq c$ and $r \in (s_{n-1}, s_n]$, suppose that ψ is a zero mode of D_r with unit L^2-norm. Regard $\check{\chi}\psi$ as defined on $\check{Y}$, and let $\check{\psi}^{\mathrm{err}} = \check{\chi}\psi - \check{\mathrm{pr}}_r(\check{\chi}\psi)$. Then*

$$\int_{\check{Y}} |\check{\psi}^{\mathrm{err}}|^2 \leq c r^{-1} \qquad \textit{and} \qquad \int_{\check{Y}} \langle \check{\psi}^{\mathrm{err}}, \check{\eta} \rangle = 0$$

for any $\check{\eta}$ in the image of $\check{\mathrm{pr}}_r$. The assertion also holds for $\check{\chi}^o\psi$, and the untwisting (4.3) of $\tilde{\chi}\psi$ and $\tilde{\chi}^o\psi$ on $\tilde{Y}$ with the projection $\tilde{\mathrm{pr}}_r$.

Proof of lemma 8.3. The orthogonality between $\check{\psi}^{\mathrm{err}}$ and $\check{\eta}$ follows directly from the construction.

With the spectral decomposition given by $\check{D}_r$, let

- $\check{\psi}^{\mathrm{err}}_{\mathrm{L}}$ be the L^2-orthogonal projection of $\check{\psi}^{\mathrm{err}}$ onto the subspace spanned by the eigensections whose eigenvalue λ satisfies $|\lambda| \geq \sqrt{\frac{r}{2}}$;
- $\check{\psi}^{\mathrm{err}}_{\mathrm{M}}$ be the L^2-orthogonal projection of $\check{\psi}^{\mathrm{err}}$ onto the subspace spanned by the eigensections whose eigenvalue λ satisfies $\frac{1}{10V} \leq |\lambda| < \sqrt{\frac{r}{2}}$, and does not arise from $\check{E}_n$;
- $\check{\psi}^{\mathrm{err}}_{\mathrm{S}}$ be the L^2-orthogonal projection of $\check{\psi}^{\mathrm{err}}$ onto the subspace spanned by the eigensections whose eigenvalue λ satisfies $\frac{1}{s_n} \leq |\lambda| < \frac{1}{10V}$, and does not arise from $\check{E}_n$;

then $\check{\chi}\psi = \check{\mathrm{pr}}_r(\check{\chi}\psi) + \check{\psi}^{\mathrm{err}}_{\mathrm{L}} + \check{\psi}^{\mathrm{err}}_{\mathrm{M}} + \check{\psi}^{\mathrm{err}}_{\mathrm{S}}$. It is an orthogonal decomposition with respect to the L^2-inner product, and $\check{D}_r$ preserves the orthogonality. We are going to estimate the size of those three error components. Note that

$$\check{D}_r(\check{\chi}\psi) = \check{\chi}'\mathrm{cl}(\mathrm{d}\rho)\psi. \tag{8.5}$$

The function $\check{\chi}'$ only supports on where $1+8\epsilon \leq \rho \leq 1+10\epsilon$, and the Clifford action of $\mathrm{d}\rho$ switches the two components of ψ, see (4.1).

The component with large eigenvalue. The estimate on $\check{\psi}^{\mathrm{err}}_{\mathrm{L}}$ is easy to come by. With (8.5), there exists a constant c_8 such that

$$\int_{\check{Y}} |\check{\psi}^{\mathrm{err}}_{\mathrm{L}}|^2 \leq 2r^{-1} \int_{\check{Y}} |\check{D}_r(\check{\psi}^{\mathrm{err}}_{\mathrm{L}})|^2 \leq 2r^{-1} \int_{\check{Y}} |\check{D}_r(\check{\chi}\psi)|^2 \leq c_8 r^{-1}.$$

The component with medium eigenvalue. By lemma 5.5, all the eigenvalues λ with $|\lambda| < \sqrt{\frac{r}{2}}$ corresponds to a *unique* $\mathcal{S}_{k,m}$. Let $\check{\psi}^{\mathrm{eig}}_{k,m}$ be the corresponding eigensection with $\int_{\check{Y}} |\check{\psi}^{\mathrm{eig}}_{k,m}|^2 = 1$. By (8.5),

$$\begin{aligned}
&|\int_{\check{Y}} \langle \check{\chi}\psi, \check{\psi}^{\mathrm{eig}}_{k,m}\rangle|^2 \\
&= \frac{1}{\lambda^2} |\int_{\check{Y}} \langle (\check{\chi})'\mathrm{cl}(\mathrm{d}\rho)\psi, \check{\psi}^{\mathrm{eig}}_{k,m}\rangle|^2 \\
&\leq 200V^2 \big(\int_{\check{Y}} |(\check{\chi})'\beta_{k,m}|^2 \int_{\check{Y}} |\check{\alpha}^{\mathrm{eig}}_{k,m}|^2 + \int_{\check{Y}} |(\check{\chi})'\alpha_{k,m}|^2 \int_{\check{Y}} |\check{\beta}^{\mathrm{eig}}_{k,m}|^2 \big) \\
&\leq c_9 \big(\int_{\check{Y}} |(\check{\chi})'\beta_{k,m}|^2 + r^{-1} \int_{\check{Y}} |(\check{\chi})'\alpha_{k,m}|^2 \big)
\end{aligned}$$

for some constant c_9, and $\big((\check{\chi})'\alpha_{k,m}, (\check{\chi})'\beta_{k,m}\big)$ is the $\mathcal{S}_{k,m}$-component of $(\check{\chi})'\psi$. The first inequality follows from the fact that $\mathrm{cl}(\mathrm{d}\rho)$ switches the components. The second inequality follows from lemma 5.4 on $\check{\psi}^{\mathrm{eig}}_{k,m}$.

If we sum up the above inequality over those (k,m) involving in $\check{\psi}^{\mathrm{err}}_{\mathrm{M}}$,

$$\int_{\check{Y}} |\check{\psi}^{\mathrm{err}}_{\mathrm{M}}|^2 \leq c_9 \big(\int_{\check{Y}} |(\check{\chi})'\beta|^2 + r^{-1} \int_{\check{Y}} |(\check{\chi})'\alpha|^2 \big).$$

With proposition 3.1, we get the estimate on $\check{\psi}^{\mathrm{err}}_{\mathrm{M}}$.

The component with small eigenvalue. By the same token, all the eigenvalues λ with $|\lambda| < \frac{1}{10V}$ corresponds to a *unique* $\mathcal{S}_{k,m}$. Proposition 5.6 and proposition 5.10 give the approximation of the corresponding eigensections

$$\check{\psi}^{\text{eig}}_{k,m} = \check{\mathfrak{q}}_{k,m}\check{\psi}_{k,m} + \check{\psi}^{(3)}_{k,m}.$$

If $m > \frac{k}{V}$ or $m < \frac{k}{V}(1 - 11\epsilon)$, the support of $\check{\psi}_{k,m}$ and $(\check{\chi})'$ are disjoint. With (8.5), there exists a constant c_8 such that

$$\begin{aligned} |\int_{\check{Y}} \langle \check{\chi}\psi, \check{\psi}^{\text{eig}}_{k,m}\rangle|^2 &= \frac{1}{\lambda^2}|\int_{\check{Y}} \langle (\check{\chi})'\text{cl}(\text{d}\rho)\psi, \check{\psi}^{(3)}_{k,m}\rangle|^2 \\ &\leq c_{10}s_n^2 r^{-3}\int_{\check{Y}} |(\check{\chi})'\psi_{k,m}|^2 < 10c_{10}r^{-1}\int_{\check{Y}} |(\check{\chi})'\psi_{k,m}|^2 \end{aligned}$$

for all $r \geq c_{10}$.

If $\frac{k}{V}(1 - 11\epsilon) \leq m \leq \frac{k}{V}$, lemma 5.15 with the condition that $|\lambda| < \frac{1}{10V}$ implies that $k = n$. However, lemma 5.15 also implies that such (k, m) belongs to $\check{E}$. Therefore, these (k, m) do not involve in $\check{\psi}^{\text{err}}_{\text{S}}$.

By summing up the above inequality over those (k, m) involving in $\check{\psi}^{\text{err}}_{\text{S}}$, we obtain the estimate on $\check{\psi}^{\text{err}}_{\text{S}}$.

It is clear that the assertion also holds for $\check{\chi}^o\psi$. Note that we can project $\check{\chi}^o\psi$ onto a smaller space, and we will come to this point later.

The discussion for $\tilde{\chi}\psi$ and $\tilde{\chi}^o\psi$ are parallel to the above argument, and will be omitted. □

From now on, when we talk about $\tilde{\chi}\psi$ on $\tilde{Y}$, we will implicitly apply the untwisting (4.3) on it. On the other hand, when we regard some section on $\tilde{Y}$ as defined on Y, we will implicitly undo the untwisting.

Step 5. For any zero mode ψ of D_r, consider $\hat{\text{pr}}_n(\hat{\chi}\psi)$ where $\hat{\text{pr}}_n$ is the composition of the following three projection

$$\hat{\chi}\psi \longmapsto \hat{\chi}\alpha \longmapsto \hat{\chi}\alpha_n e^{in\phi}(2\pi V)^{-\frac{1}{2}} \longmapsto \hat{\text{pr}}_n(\hat{\chi}\alpha_n)e^{in\phi}(2\pi V)^{-\frac{1}{2}}.$$

The first map is the projection onto the first component. The second map is the projection onto the frequency n component with respect to the S^1-action in $e^{i\psi}$, and $(2\pi V)^{-\frac{1}{2}}$ is just a normalizing constant, see subsection 4.4. The last map is the L^2-orthogonal projection onto the kernel of $\bar{\partial}_n$ as discussed in section 7. The last projection and the composition of the three projections share the same notation, but it should not cause any confusion.

Lemma 8.4. *There exists a constant c which has the following significance: For any $n \geq c$ and $r \in (s_{n-1}, s_n]$, suppose that ψ is a zero mode of D_r with unit L^2-norm. Regard $\hat{\chi}\psi$ as defined on $\Sigma \times S^1$, and let $\hat{\psi}^{\text{err}} = \hat{\chi}\psi - \hat{\text{pr}}_n(\hat{\chi}\psi)$. Then*

$$\int_{\Sigma\times S^1} |\hat{\psi}^{\text{err}}|^2 \leq cr^{-1}, \qquad \textit{and} \qquad \int_{\Sigma\times S^1} \langle \hat{\psi}^{\text{err}}, \hat{\eta}\rangle = 0$$

for any $\hat{\eta}$ in the image of $\hat{\text{pr}}_n$. The assertion also holds for $\hat{\chi}^o\psi$.

Proof of lemma 8.4. The orthogonality between $\hat{\psi}^{\mathrm{err}}$ and $\hat{\eta}$ follows directly from the construction.

To estimate the size of $\hat{\psi}^{\mathrm{err}}$, consider the Fourier expansion of $\hat{\chi}\psi$:

$$\hat{\chi}\alpha = \sum_{\mathfrak{n}\in\mathbb{Z}} \hat{\chi}\alpha_{\mathfrak{n}} e^{i\mathfrak{n}\phi}(2\pi V)^{-\frac{1}{2}} \quad \text{and} \quad \hat{\chi}\beta = \sum_{\mathfrak{n}\in\mathbb{Z}} \hat{\chi}\beta_{\mathfrak{n}} e^{i(\mathfrak{n}+1)\phi}(2\pi V)^{-\frac{1}{2}}, \text{ then}$$

$$\int_{\Sigma\times S^1} |\hat{\chi}\alpha|^2 = \sum_{\mathfrak{n}\in\mathbb{Z}} \int_{\Sigma} |\hat{\chi}\alpha_{\mathfrak{n}}|^2 \qquad \text{and} \qquad \int_{\Sigma\times S^1} |\hat{\chi}\beta|^2 = \sum_{\mathfrak{n}\in\mathbb{Z}} \int_{\Sigma} |\hat{\chi}\beta_{\mathfrak{n}}|^2.$$

As in section 7, let $\mathcal{D}_{r,\mathfrak{n}}$ be the operator (4.8), with n replaced by $\mathfrak{n}$. Since $D_r(\hat{\chi}\psi) = \mathrm{cl}(\mathrm{d}\hat{\chi})\psi$, $\mathcal{D}_{r,\mathfrak{n}}(\hat{\chi}\alpha_{\mathfrak{n}}, \hat{\chi}\beta_{\mathfrak{n}})$ only supports on $\Sigma\backslash\Sigma_{-6\epsilon}$, and

$$\mathcal{D}_{r,\mathfrak{n}}(\hat{\chi}\alpha_{\mathfrak{n}}, \hat{\chi}\beta_{\mathfrak{n}}) = \big(-(\hat{\chi})'\beta_{\mathfrak{n}}, (\hat{\chi})'\alpha_{\mathfrak{n}}\big). \tag{8.6}$$

The main task is to estimate $\hat{\chi}\alpha_{\mathfrak{n}}$. We divide the argument into three cases according to the value of $\mathfrak{n}$. Remember that $\gamma_{\mathfrak{n}} = \frac{2\mathfrak{n}}{V}$.

Case 1: $|\gamma_{\mathfrak{n}} - r| \geq r^{\frac{1}{2}}$. With lemma 7.1 and (8.6), there exists a constant c_9 such that

$$\int_{\Sigma} |\hat{\chi}\alpha_{\mathfrak{n}}|^2 + |\hat{\chi}\beta_{\mathfrak{n}}|^2 \leq c_{11} r^{-1}\big(\int_{\Sigma\backslash\Sigma_{-6\epsilon}} |\alpha_{\mathfrak{n}}|^2 + |\beta_{\mathfrak{n}}|^2\big)$$

for all $r \geq c_{11}$.

Case 2: $|\gamma_{\mathfrak{n}} - r| < r^{\frac{1}{2}}$ and the $(\ker\bar{\partial}_{\mathfrak{n}})^{\perp}$-component. With the first inequality of proposition 7.2 and (8.6), there exists a constant c_{12} such that

$$\int_{\Sigma} |\hat{\chi}\alpha_{\mathfrak{n}} - \hat{\mathrm{pr}}_{\mathfrak{n}}(\hat{\chi}\alpha_{\mathfrak{n}})|^2 + |\hat{\chi}\beta_{\mathfrak{n}}|^2 \leq c_{12} r^{-1}\big(\int_{\Sigma\backslash\Sigma_{-6\epsilon}} |\alpha_{\mathfrak{n}}|^2 + |\beta_{\mathfrak{n}}|^2\big)$$

for all $r \geq c_{12}$.

Case 3: $\mathfrak{n} \neq n$, $|\gamma_{\mathfrak{n}} - r| < r^{\frac{1}{2}}$ and the $\ker\bar{\partial}_{\mathfrak{n}}$-component. For any $\mathfrak{n} \neq n$, $|r - \gamma_{\mathfrak{n}}| \geq \frac{1}{10V}$. With the second inequality of proposition 7.2 and (8.6), there exists a constant c_{13} such that

$$\int_{\Sigma} |\hat{\mathrm{pr}}_{\mathfrak{n}}(\hat{\chi}\alpha_{\mathfrak{n}})|^2 \leq c_{13} \int_{\Sigma\backslash\Sigma_{-6\epsilon}} |\beta_{\mathfrak{n}}|^2$$

for all $r \geq c_{13}$ and $\mathfrak{n} \neq n$.

If we sum up the conclusion of these three cases, we have

$$\int_{\Sigma\times S^1} |\hat{\psi}^{\mathrm{err}}|^2 \leq \int_{(\Sigma\backslash\Sigma_{-6\epsilon})\times S^1} \big((c_{11} + c_{12}) r^{-1}|\psi|^2 + c_{13}|\beta|^2\big).$$

With proposition 3.1, we complete the proof of lemma 8.4. For $\hat{\chi}^o\psi$, the proof is the same. $\square$

Step 6. This step projects the zero modes onto a vector space spanned by certain almost eigensections. The projection will depend on r. In other words, the zero modes are *not* projected onto the *same* vector space.

Proposition 8.5. *There exists a constant c which has the following significance: For any $n \geq c$ and $r \in (s_{n-1}, s_n]$, suppose that ψ is a zero mode of D_r with unit L^2-norm. Let*

$$\Pi_r(\psi) = \hat{\chi}\hat{\mathrm{pr}}_n(\hat{\chi}^o\psi) + \check{\chi}\check{\mathrm{pr}}_r(\check{\chi}^o\psi) + \tilde{\chi}\tilde{\mathrm{pr}}_r(\tilde{\chi}^o\psi).$$

Then

(i) $\int_Y |\Pi_r(\psi)|^2 \geq 1 - cs_n^{-1}$.

(ii) *If there are two such zero modes ψ_1 and ψ_2 at r_1 and r_2 with $s_{n-1} < r_1 < r_2 \leq s_n$, then*

$$\Big|\int_Y \langle \Pi_{r_1}(\psi_1), \Pi_{r_2}(\psi_2)\rangle\Big| \leq cs_n^{-1}$$

(iii) *If there are two such zero modes ψ_1 and ψ_2 both at $r \in (s_{n-1}, s_n]$, and $\int_Y \langle \psi_1, \psi_2\rangle = 0$, then*

$$\Big|\int_Y \langle \Pi_{r_1}(\psi_1), \Pi_{r_2}(\psi_2)\rangle\Big| \leq cs_n^{-1}$$

Proof of proposition 8.5. We start with assertion (ii). Suppose that there are two such zero modes $\{\psi_j\}_{j=1,2}$. Apply lemma 8.4 on $\hat{\chi}^o\psi_j$, and lemma 8.3 on $\check{\chi}^o\psi_j$ and $\tilde{\chi}^o\psi_j$, then multiply them by the cut-off functions with larger support. We have

$$\begin{aligned}
\hat{\chi}^o\psi_j &= \hat{\chi}\hat{\chi}^o\psi_j = \hat{\chi}\hat{\mathrm{pr}}_n(\hat{\chi}^o\psi_j) + \hat{\chi}\hat{\psi}_j^{\mathrm{err}_o}, \\
\check{\chi}^o\psi_j &= \check{\chi}\check{\chi}^o\psi_j = \check{\chi}\check{\mathrm{pr}}_r(\check{\chi}^o\psi_j) + \check{\chi}\check{\psi}_j^{\mathrm{err}_o}, \\
\tilde{\chi}^o\psi_j &= \tilde{\chi}\tilde{\chi}^o\psi_j = \tilde{\chi}\tilde{\mathrm{pr}}_r(\tilde{\chi}^o\psi_j) + \tilde{\chi}\tilde{\psi}_j^{\mathrm{err}_o},
\end{aligned}$$

and all terms can be regarded as defined on Y. Also apply lemma 8.4 on $\hat{\chi}\psi_j$, and lemma 8.3 on $\check{\chi}\psi_j$ and $\tilde{\chi}\psi_j$. We have $\hat{\chi}\psi_j = \hat{\mathrm{pr}}_n(\hat{\chi}\psi_j) + \hat{\psi}_j^{\mathrm{err}}$, $\check{\chi}\psi_j = \check{\mathrm{pr}}_r(\check{\chi}\psi_j) + \check{\psi}_j^{\mathrm{err}}$ and $\tilde{\chi}\psi_j = \tilde{\mathrm{pr}}_r(\tilde{\chi}\psi_j) + \tilde{\psi}_j^{\mathrm{err}}$. Not all these terms can be regarded as defined on Y.

The inner product between $\Pi_{r_1}(\psi_1)$ and $\Pi_{r_2}(\psi_2)$ is equal to

$$\int_Y \langle \psi_1 - \hat{\chi}\hat{\psi}_1^{\mathrm{err}_o} - \check{\chi}\check{\psi}_1^{\mathrm{err}_o} - \hat{\chi}\hat{\psi}_1^{\mathrm{err}_o}, \psi_2 - \hat{\chi}\hat{\psi}_2^{\mathrm{err}_o} - \check{\chi}\check{\psi}_2^{\mathrm{err}_o} - \hat{\chi}\hat{\psi}_2^{\mathrm{err}_o}\rangle.$$

We would like to show that the magnitude of all these sixteen pairings is less than $c_{14}s_n^{-1}$ for some constant c_{14}. There are four types of these pairings.

Type 1: $\int_Y \langle \psi_1, \psi_2\rangle$. Proposition 3.3 gives the estimate on this term.

Type 2: the pairings between ψ_j and the error term on $\Sigma \times S^1$. By lemma 8.4,

$$\begin{aligned}\int_Y \langle \psi_1, \hat{\chi}\hat{\psi}_2^{\mathrm{err}_o}\rangle &= \int_{\Sigma\times S^1}\langle \hat{\chi}\psi_1, \hat{\psi}_2^{\mathrm{err}_o}\rangle = \int_{\Sigma\times S^1}\langle \hat{\mathrm{pr}}_n(\hat{\chi}\psi_1) + \hat{\psi}_1^{\mathrm{err}}, \hat{\psi}_2^{\mathrm{err}_o}\rangle \\ &= \int_{\Sigma\times S^1}\langle \hat{\psi}_1^{\mathrm{err}}, \hat{\psi}_2^{\mathrm{err}_o}\rangle.\end{aligned}$$

We conclude that $\big|\int_Y \langle \psi_1, \hat{\chi}\hat{\psi}_2^{\mathrm{err}_o}\rangle\big| \leq c_{15}(r_1 r_2)^{-\frac{1}{2}} \leq 2c_{15}s_n^{-1}$ for some constant c_{15}.

Type 3: the pairings between ψ_j and the error term on $\check{Y}$ or $\tilde{Y}$. Similarly,

$$\int_Y \langle \psi_1, \check{\chi}\check{\psi}_2^{\mathrm{err}_o}\rangle = \int_{\check{Y}}\langle \check{\mathrm{pr}}_{r_1}(\check{\chi}\psi_1), \check{\psi}_2^{\mathrm{err}_o}\rangle + \langle \check{\psi}_1^{\mathrm{err}}, \check{\psi}_2^{\mathrm{err}_o}\rangle.$$

Lemma 8.3 implies that the second term $\big|\int_{\check{Y}}\langle \check{\psi}_1^{\mathrm{err}}, \check{\psi}_2^{\mathrm{err}_o}\rangle\big| \leq 2c_{16}s_n^{-1}$ for some constant c_{16}. Unlike type 2, the first pairing might not be zero. However, with the help of corollary 5.7, corollary 5.11, the triangle inequality and the Cauchy–Schwarz inequality, there exists a constant c_{17} such that

$$\Big|\int_{\check{Y}}\langle \check{\mathrm{pr}}_{r_1}(\check{\chi}\psi_1), \check{\psi}_2^{\mathrm{err}_o}\rangle\Big| \leq c_{17}s_n^{-\frac{3}{2}}(\#\check{E}_n)\Big(\int_{\check{Y}}|\check{\psi}_2^{\mathrm{err}_o}|\Big)^{\frac{1}{2}}.$$

For the same reason as (8.2), $\check{E}_n$ is less than some multiple of s_n, and $\check{\psi}_2^{\mathrm{err}_o}$ is estimated by lemma 8.3. Hence, there exists a constant c_{18} such that $\big|\int_{\check{Y}}\langle \check{\mathrm{pr}}_{r_1}(\check{\chi}\psi_1), \check{\psi}_2^{\mathrm{err}_o}\rangle\big| \leq c_{18}s_n^{-1}$.

Type 4: the pairings between two error terms. Lemma 8.3, lemma 8.4 and the Cauchy–Schwarz inequality give the estimate on those pairings.

The proof for assertion (i) and (iii) is the same, up to a minor modification for type 1. We complete the proof of proposition 8.5. □

Step 7. This step is a digression on the r-dependence of the eigensection approximation constructed in subsection 5.2. Let us start with $\check{Y}$. Suppose that $\check{D}_r$ on $\mathcal{S}_{k,m}$ has an eigenvalue λ_0 with $|\lambda_0| \leq 1$, and proposition 5.6 and proposition 5.10 apply.

Case 1. When $|m| < (\frac{1}{32\epsilon^2} - 1)k$, the main term $\check{\psi}_{k,m}$ in proposition 5.6 consists of the zeroth, first and second order terms, see (5.25). Let $\check{\psi}_{k,m}^{(0)}$ be the section which consists of only the zeroth and first order term of $\check{\psi}_{k,m}$. Namely, throw away the $\mathfrak{a}_2(x,r)$ and $\mathfrak{b}_2(x)$ terms in (5.25). The key feature of these sections $\check{\psi}_{k,m}^{(0)}$ is that they are independent of r. Let $\check{\psi}_{k,m}^{(2)}$ be $\check{\psi}_{k,m} - \check{\mathfrak{q}}_{k,m}\check{\psi}_{k,m}^{(0)}$. Namely, $\check{\psi}_{k,m}^{(2)}$ is the sum of the second order term in $\check{\mathfrak{q}}_{k,m}\check{\psi}_{k,m}$ and $\check{\psi}_{k,m}^{(3)}$. Under the assumption of proposition 5.6, it is easy to

see that there exists a constant c_{19} such that

$$\int_{\check{Y}} |\check{\psi}^{(2)}_{k,m}|^2 \leq c_{19} r^{-2}.$$

The coefficients $\check{\mathfrak{q}}_{k,m}$ in proposition 5.6 also depends on r, but they are only scalars. Note that $\check{\psi}^{(0)}_{k,m}$ can be regarded as a section on Y.

According to the discussion in subsection 5.4, when $\frac{k}{V}(1-11\epsilon) \leq m \leq \frac{k}{V}$, the first and second order term are zero, and $\check{\psi}^{o}_{k,m} = \check{\psi}_{k,m}$ and $\int_{\check{Y}} |\check{\psi}^{(2)}_{k,m}|^2 \leq c_{20}\exp(-\frac{r}{c_{20}})$. Moreover, for $k = n$, the first component of $\check{\psi}^{(0)}_{k,m}$ is the same as $\check{\zeta}_{n,m} e^{in\phi}(2\pi V)^{-\frac{1}{2}}$ given by (7.3), and second component of $\check{\psi}^{(0)}_{k,m}$ is zero.

Case 2. When $|m| \geq (\frac{1}{32\epsilon^2} - 1)k$, the main term $\check{\psi}_{k,m}$ in proposition 5.10 is already independent of r. To unify the notation, let $\check{\psi}^{(0)}_{k,m} = \check{\psi}_{k,m}$ and $\check{\psi}^{(2)}_{k,m} = \check{\psi}^{(3)}_{k,m}$.

On $\tilde{Y}$. The discussion on the r-dependence is the same. We just mention the result on the special regions discussed in subsection 5.4. When

$$\begin{cases} k + mh(1) = n \ \text{ and } \ \frac{2n}{V}(v - 1 - 11\epsilon) \leq m \leq \frac{2n}{V}(v-1), \text{ or} \\ k + mh(-1) = n \ \text{ and } \ \frac{2n}{V}(v+1) \leq m \leq \frac{2n}{V}(v + 1 + 11\epsilon), \end{cases}$$

the first component of $\tilde{\psi}^{(0)}_{k,m}$ is the same as the untwisting of $\tilde{\zeta}_{n,m} e^{in\phi}(2\pi V)^{-\frac{1}{2}}$ given by (7.4), and second component of $\tilde{\psi}^{(0)}_{k,m}$ is zero. The remainder term satisfies $\int_{\tilde{Y}} |\tilde{\psi}^{(2)}_{k,m}|^2 \leq c_{20}\exp(-\frac{r}{c_{20}})$.

Step 8. In this step, we are going to throw away the r-dependent part of $\check{\chi}\mathrm{pr}_r(\check{\chi}^o\psi)$ and $\tilde{\chi}\mathrm{pr}_r(\tilde{\chi}^o\psi)$.

Lemma 8.6. *There exists a constant c which has the following significance: For any $n \geq c$ and $r \in (s_{n-1}, s_n]$, suppose that ψ is a zero mode of D_r with unit L^2-nrom. Then there exists a section $\check{\psi}^{\mathrm{rem}}$ such that*

(i) *The support of $\check{\psi}^{\mathrm{rem}}$ is contained in the support of $\check{\chi}$. Hence, $\check{\psi}^{\mathrm{rem}}$ can be regarded as defined either on Y or $\check{Y}$.*

(ii) $\int_Y |\check{\psi}^{\mathrm{rem}}|^2 \leq cs_n^{-2}$.

(iii) $\check{\chi}\mathrm{pr}_r(\check{\chi}^o\psi) - \check{\psi}^{\mathrm{rem}}$ *belongs to the vector space spanned by*

$$\{\check{\psi}^{(0)}_{k,m} | (k,m) \in \check{E}^o_n\} \oplus \{\check{\psi}^{(0)}_{k,m} | k = n, \text{ and } \frac{n}{V}(1-7\epsilon) \leq m < \frac{n}{V}\}$$

Proof of lemma 8.6. For each $(k,m) \in \check{E}_n$, $\check{D}_r$ has a unique eigenvalue λ_0 on $\mathcal{S}_{k,m}$ with $|\lambda_0| \leq 1$. Let $\check{\psi}^{\mathrm{eig}}_{k,m}$ be the corresponding eigensection given by proposition 5.6 and proposition 5.10. By the definition of $\check{\mathrm{pr}}_r$ in step 4, we have

$$\check{\mathrm{pr}}_r(\check{\chi}^o\psi) = \sum_{(k,m)\in\check{E}_n} \check{\mathfrak{c}}_{k,m}\check{\psi}^{\mathrm{eig}}_{k,m}$$

where $\check{\mathfrak{c}}_{k,m} = \int_{\check{Y}} \langle \check{\chi}^o \psi, \check{\psi}^{\mathrm{eig}}_{k,m} \rangle$.

Remember that $\check{E}_n \backslash \check{E}^o_n$ has a complete characterization by (8.4). Let $\check{\psi}^{\mathrm{rem}}$ be the sum of the following terms:

- $\check{\chi} \check{\mathfrak{c}}_{k,m} \check{\psi}^{(2)}_{k,m}$ for $(k,m) \in \check{E}^o_n$.
- $\check{\chi} \check{\mathfrak{c}}_{k,m} \check{\psi}^{(2)}_{k,m}$ for $(k,m) \in \check{E}_n \backslash \check{E}^o_n$ and $m \geq \frac{n}{V}(1-7\epsilon)$.
- $\check{\chi} \check{\mathfrak{c}}_{k,m} \check{\psi}^{\mathrm{eig}}_{k,m}$ for $(k,m) \in \check{E}_n \backslash \check{E}^o_n$ and $m < \frac{n}{V}(1-7\epsilon)$.

We now estimate their L^2-norm.

- For the terms of the first and second kind, step 7 finds a constant c_{21} such that
$$\int_{\check{Y}} |\check{\chi} \check{\mathfrak{c}}_{k,m} \check{\psi}^{(2)}_{k,m}|^2 \leq c_{21} r^{-2} \int_{\check{Y}} |(\check{\chi}^o \psi)_{k,m}|^2$$
where $(\check{\chi}^o \psi)_{k,m}$ is the $\mathcal{S}_{k,m}$-component of $\check{\chi}^o \psi$.
- For the terms of the third kind,
$$\begin{aligned}\int_{\check{Y}} |\check{\chi} \check{\mathfrak{c}}_{k,m} \check{\psi}^{\mathrm{eig}}_{k,m}|^2 \leq |\check{\mathfrak{c}}_{k,m}|^2 &= \Big| \int_{\check{Y}} \langle \check{\chi}^o \psi, \check{\psi}^{\mathrm{eig}}_{k,m} \rangle \Big|^2 \\ &= \Big| \int_{\check{Y}} \langle (\check{\chi}^o \psi)_{k,m}, \check{\mathfrak{q}}_{k,m} \check{\psi}^{(0)}_{k,m} + \check{\psi}^{(2)}_{k,m} \rangle \Big|^2 \\ &\leq c_{20} \exp(-\frac{r}{c_{20}}) \int_{\check{Y}} |(\check{\chi}^o \psi)_{k,m}|^2.\end{aligned}$$

  For the last inequality, note that the support of $\check{\chi}^o$ and $\check{\psi}_{k,m}$ are disjoint, and the L^2-norm of $\check{\psi}^{(2)}_{k,m}$ is exponentially small by step 7.

By summing up the above inequalities, we prove the assertion (ii) of the lemma. Assertion (i) is follows from the construction of $\check{\psi}^{\mathrm{rem}}$.

For the last assertion, it is clear that $\check{\chi} \mathrm{pr}_r(\check{\chi}^o \psi) - \check{\psi}^{\mathrm{rem}}$ belongs to the vector space spanned by those elements in assertion (iii), up to the multiplication by $\check{\chi}$. However, $\check{\chi}$ is equal to 1 on the support of those elements. Therefore, we complete the proof of lemma 8.6. $\square$

The argument for the Dehn-twist region is the same. We just state the result.

Lemma 8.7. *There exists a constant c which has the following significance: For any $n \geq c$ and $r \in (s_{n-1}, s_n]$, suppose that ψ is a zero mode of D_r with unit L^2-norm. Then there exists a section $\tilde{\psi}^{\mathrm{rem}}$ such that*

(i) *The support of $\tilde{\psi}^{\mathrm{rem}}$ is contained in the support of $\tilde{\chi}$. Hence, up to the untwisting (4.3), $\tilde{\psi}^{\mathrm{rem}}$ can be regarded as defined either on Y or $\tilde{Y}$.*

(ii) $\int_{\tilde{Y}} |\tilde{\psi}^{\mathrm{rem}}|^2 \leq c s_n^{-2}$.

(iii) $\tilde{\chi}\mathrm{pr}_r(\tilde{\chi}^o\psi) - \tilde{\psi}^{\mathrm{rem}}$ *belongs to the vector space spanned by*

$$\big\{\tilde{\psi}^{(0)}_{k,m}\big|(k,m)\in\tilde{E}^o_n\big\}$$
$$\oplus\,\big\{\tilde{\psi}^{(0)}_{k,m}\big|k+mh(1)=n,\ \textit{and}\ \tfrac{2n}{V}(v-1-7\epsilon)\le m<\tfrac{2n}{V}(v-1)\big\}$$
$$\oplus\,\big\{\tilde{\psi}^{(0)}_{k,m}\big|k+mh(1)=n,\ \textit{and}\ \tfrac{2n}{V}(v+1)<m\le\tfrac{2n}{V}(v+1+7\epsilon)\big\}$$

Step 9. In this step, we apply lemma 7.3 to study $\hat{\chi}\mathrm{pr}_n(\hat{\chi}^o\psi)$.

Lemma 8.8. *There exists a constant c which has the following significance: For any $n\ge c$ and $r\in(s_{n-1},s_n]$, suppose that ψ is a zero mode of D_r with unit L^2-norm. Then there exists a section $\hat{\psi}^{\mathrm{rem}}$ such that*

(i) *The support of $\hat{\psi}^{\mathrm{rem}}$ is contained in the support of $\hat{\chi}$.*

(ii) $\int_Y|\hat{\psi}^{\mathrm{rem}}|^2\le cs_n^{-2}$.

(iii) *The second component of $\hat{\chi}\mathrm{pr}_n(\hat{\chi}^o\psi)-\hat{\psi}^{\mathrm{rem}}$ is zero. The first component of $\hat{\chi}\mathrm{pr}_n(\hat{\chi}^o\psi)-\hat{\psi}^{\mathrm{rem}}$ belongs to the vector space spanned by*

$$\Big\{\hat{\chi}e^{in\phi}\cdot\ker_0\bar{\partial}_n\Big\}\oplus\Big\{\check{\zeta}_{n,m}e^{in\phi}\big|\tfrac{n}{C}(1-11\epsilon)\le m\le\tfrac{n}{C}(1-3\epsilon)\Big\}$$
$$\oplus\Big\{\tilde{\zeta}_{n,m}e^{in\phi}\big|\tfrac{2n}{V}(v+1+3\epsilon)\le m\le\tfrac{2n}{V}(v+1+11\epsilon),\ \textit{or}$$
$$\tfrac{2n}{V}(v-1-11\epsilon)\le m\le\tfrac{2n}{V}(v-1-3\epsilon)\Big\}.$$

Proof of lemma 8.8. According to the decomposition given by lemma 7.3, the first component of $\mathrm{pr}_n(\hat{\chi}^o\psi)$ can be expressed as

$$\zeta_0+\sum\check{\mathfrak{c}}_{n,m}(\check{\mathfrak{p}}_{n,m}\check{\zeta}_{n,m}+\check{\zeta}^{\mathrm{rem}}_{n,m})+\sum\tilde{\mathfrak{c}}_{n,m}(\tilde{\mathfrak{p}}_{n,m}\tilde{\zeta}_{n,m}+\tilde{\zeta}^{\mathrm{rem}}_{n,m})$$

where $\zeta_0\in\ker_0\bar{\partial}_n$. Let $\hat{\psi}^{\mathrm{rem}}$ be the sum of the following terms:

- $\hat{\chi}\check{\mathfrak{c}}_{n,m}(\check{\mathfrak{p}}_{n,m}\check{\zeta}_{n,m}+\check{\zeta}^{\mathrm{rem}}_{n,m})$ for $\frac{n}{V}(1-3\epsilon)<m<\frac{n}{V}$.
- $\hat{\chi}\check{\mathfrak{c}}_{n,m}\check{\zeta}^{\mathrm{rem}}_{n,m}$ for $\frac{n}{V}(1-11\epsilon)\le m\le\frac{n}{V}(1-3\epsilon)$.
- $\hat{\chi}\tilde{\mathfrak{c}}_{n,m}(\tilde{\mathfrak{p}}_{n,m}\tilde{\zeta}_{n,m}+\tilde{\zeta}^{\mathrm{rem}}_{n,m})$ for $\frac{2n}{V}(v+1)<m<\frac{2n}{V}(v+1+3\epsilon)$ or $\frac{2n}{V}(v-1-3\epsilon)<m<\frac{2n}{V}(v-1)$.
- $\hat{\chi}\check{\mathfrak{c}}_{n,m}\tilde{\zeta}^{\mathrm{rem}}_{n,m}$ for $\frac{2n}{V}(v+1+3\epsilon)\le m\le\frac{2n}{V}(v+1+11\epsilon)$ or $\frac{2n}{V}(v-1-11\epsilon)\le m\le\frac{2n}{V}(v-1-3\epsilon)$.

For the terms of the first and third kind, lemma 7.3 finds a constant c_{22} such that their L^2-norm is less than $c_{22}\exp(-\frac{n}{c_{22}})$. For the terms of the second and forth kind, the supports of $\check{\zeta}_{n,m}$ and $\tilde{\zeta}_{n,m}$ are disjoint from the support of $\hat{\chi}^o$. A similar argument as in the proof of lemma 8.6 shows that their L^2-norm is less than $c_{22}\exp(-\frac{n}{c_{22}})$. With the triangle inequality,

$$\int_Y|\hat{\psi}^{\mathrm{rem}}|^2\le\Big(\frac{110\epsilon}{V}n\,c_{22}\exp(-\frac{n}{c_{22}})\Big)^2\le c_{23}s_n^{-2}$$

for some constant c_{23}. This proves assertion (i) and (ii) of the lemma.

For the last assertion, it is clear that $\hat{\chi}\mathrm{pr}_n(\hat{\chi}^o\psi) - \hat{\psi}^{(2)}$ belongs to the vector space spanned by those elements in assertion (iii), up to the multiplication by $\hat{\chi}$. However, for those $\check{\zeta}_{n,m}$ and $\tilde{\zeta}_{n,m}$ in the last two summand, $\hat{\chi}$ is equal to 1 on their supports. Hence, we complete the proof of lemma 8.8. □

Step 10. This is the last step. We are going to combine all the results to prove the claim (8.3).

Proposition 8.9. *There exists a constant c which has the following significance: For any $n \geq c$ and $r \in (s_{n-1}, s_n]$, suppose that ψ is a zero mode of D_r of unit L^2-norm. With proposition 8.5 and lemma 8.6, 8.7 and 8.8, let*

$$\Pi(\psi) = \Pi_r(\psi) - \hat{\psi}^{\mathrm{rem}} - \check{\psi}^{\mathrm{rem}} - \tilde{\psi}^{\mathrm{rem}}.$$

Then

(i) $\int_Y |\Pi(\psi)|^2 \geq 1 - cs_n^{-1}$.

(ii) $\Pi(\psi)$ *belongs to a vector space whose dimension is*

$$I_\Sigma(n) + \#\check{E}_n^o + \#\tilde{E}_n^o.$$

(iii) *If there are two such zero modes ψ_1 and ψ_2 at r_1 and r_2 with $s_{n-1} < r_1 < r_2 \leq s_n$, then*

$$\Big|\int_Y \langle \Pi(\psi_1), \Pi(\psi_2)\rangle\Big| \leq cs_n^{-1}$$

(iv) *If there are two such zero modes ψ_1 and ψ_2 both at $r \in (s_{n-1}, s_n]$, and $\int_Y \langle \psi_1, \psi_2 \rangle = 0$, then*

$$\Big|\int_Y \langle \Pi(\psi_1), \Pi(\psi_2)\rangle\Big| \leq cs_n^{-1}$$

Proof of proposition 8.9. Assertion (i), (iii) and (iv) follows from proposition 8.5 and lemma 8.6, 8.7 and 8.8.

With the observation in step 7, $\Pi(\psi)$ belongs to the vector space spanned by

$$\begin{aligned}
&\left\{\hat{\chi}e^{in\phi} \cdot \ker_0 \bar{\partial}_n\right\} \oplus \left\{\check{\zeta}_{n,m}e^{in\phi} \Big| \tfrac{n}{V}(1 - 11\epsilon) \leq m < \tfrac{n}{V}\right\} \\
&\oplus \Big\{\tilde{\zeta}_{n,m}e^{in\phi} \Big| \tfrac{2n}{V}(v+1) < m \leq \tfrac{2n}{V}(v + 1 + 11\epsilon), \text{ or} \\
&\qquad \tfrac{2n}{V}(v - 1 - 11\epsilon) \leq m < \tfrac{2n}{V}(v-1)\Big\} \\
&\oplus \left\{\check{\psi}^{(0)}_{k,m} \Big| (k,m) \in \check{E}_n^o\right\} \oplus \left\{\tilde{\psi}^{(0)}_{k,m} \Big| (k,m) \in \tilde{E}_n^o\right\}.
\end{aligned} \tag{8.7}$$

To be more precise, the elements in the first three summands are the sections whose first component is given by those elements and the second component is zero. By dimension counting, the dimension of the subspace spanned by the first three summands is $I_\Sigma(n)$. We complete the proof of proposition 8.9. □

According to lemma 4.2, (4.9) and (8.2), there exists a constant c_{25} such that

$$I_\Sigma(n) + \#\check{E}_n^o + \#\tilde{E}_n^o \le c_{25} s_n$$

for all $n \ge c_{25}$.

Let L_n be the index set $\{1, 2, \cdots, \mathrm{sf}_a(s_n) - \mathrm{sf}_a(s_{n-1})\}$. We may assume that there are only positive zero crossings. For each zero crossing happening between $(s_{n-1}, s_n]$, choose a zero mode with unit L^2-norm. If there are more than one zero crossings happening at some $r \in (s_{n-1}, s_n]$, choose orthonormal zero modes with respect to the L^2-inner product. Let $\{\psi_l\}_{l \in L_n}$ be the set of these zero modes.

By proposition 8.9, there exists a constant c_{26} such that

(i) $\int_Y |\Pi(\psi_l)|^2 \ge 1 - c_{26} s_n^{-1}$.

(ii) $\left| \int_Y \langle \Pi(\psi_l), \Pi(\psi_{l'}) \rangle \right| \le c_{26} s_n^{-1}$ for any $l \neq l'$.

(iii) $\Pi(\psi_l)$ belongs to a vector space (8.7), whose dimension is

$$I_\Sigma(n) + \#\check{E}_n^o + \#\tilde{E}_n^o \le c_{25} s_n.$$

for all $n \ge c_{26}$. After normalizing the L^2-norm of $\Pi(\psi_l)$, lemma 8.2 applies. This proves the claim (8.3). End of the proof of theorem 8.1. □

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